This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

S

AL.
S.

Y.

William Collins, Sons, & Co's Educational Works.

COLLINS' ELEMENTARY AND ADVANCED SCIENCE SERIES,

Adapted to the requirements of the South Kensington Syllabus, for Students in Science and Art Classes, and Higher and Middle Class Schools.

ELEMENTARY SERIES.

In Fcap. 8vo, fully Illustrated, cloth lettered, price 1s. each volume.

1. Practical Plane and Solid Geometry. By H. Angel, Islington Science School, London.
2. Machine Construction and Drawing. By E. Tomkins. Vol. I. Text, Vol. II. Plates.

3a Building Construction—Stone, Brick, and Slate Work. By R. S. Burn, C.E. Vol. I. Text, Vol. II. Plates.
3b Building Construction—Timber and Iron Work. By R. S. Burn, C.E. Vol. I. Text, Vol. II. Plates.
4a Naval Architecture—Laying Off. By S. J P. Thearle, F.R.S.N.A., London. Vol. I. Text, 1s. Vol. II. Plates, 2s.
4b Naval Architecture—Wood and Iron Shipbuilding. By S. J. P. Thearle. Vol. I Text, 1s. Vol. II. Plates, 2s.

5. Pure Mathematics. By L. Sergeant, B.A. (Camb.).
6. Theoretical Mechanics. By J. T. Bottomley, Glasgow Univ.
7. Applied Mechanics. By W. Rossiter, F.R.A.S.
8. Acoustics, Light, and Heat. By William Lees, A.M.
9. Magnetism and Electricity By John Angell, Manchester.
10. Inorganic Chemistry By Dr. W. B. Kemshead, F.R.A.S.
11. Organic Chemistry. By W. Marshall Watts, D.Sc.
12. Geology. By W. S. Davis, LL.D., Derby.
13. Mineralogy. By J. H. Collins, F.G.S., Falmouth.
14. Animal Physiology. By John Angell, Manchester.
15. Zoology. By M. Harbison, Newtonards.
16. Vegetable Anatomy and Physiology. By Professor Balfour.
17. Systematic and Economic Botany. By Professor Balfour.

18a Principles of Mining—Coal. By J. H. Collins, F.G.S.
18b Principles of Mining—Iron. By J. H. Collins, F.G.S.

19. Metallurgy. By John Mayer, F.C.S., Glasgow.
20. Navigation. By Henry Evers, LL.D., Plymouth.
21. Nautical Astronomy. By Henry Evers, LL.D.

22a Steam and the Steam Engine—Land and Marine. By Dr. Evers.
22b Steam Engine—Locomotive. By H. Evers, LL.D.

23. Physical Geography. By John Macturk, F.R.G.S.
24. Practical Chemistry. By John Howard, London.
25. Astronomy. By J. J. Plummer, Observatory, Durham.
26. Manual of Qualitative Chemical Analysis. By F. Beilstein.

Extra Volumes, Post 8vo, Cloth, 1s. 6d.

7b Applied Mechanics. By Henry Evers, LL.D.
9b Magnetism and Electricity. Enlarged Edition. By J. Angell, Manchester.
14b Animal Physiology. Enlarged Edition, 92 Illustrations. By J. Angell, Manchester.
15b Gener[redacted] By T. C. MacGinley.

Lon[redacted] Hill Works, Glasgow.

William Collins, Sons, & Co.'s Educational Works.

ADVANCED SCIENCE SERIES,

Adapted to the requirements of the South Kensington Syllabus, for Students in Science and Art Classes, and Higher and Middle Class Schools.

In the Press, and in Preparation, Post 8vo, fully Illustrated, cloth lettered, price 2s. 6d. each volume.

1. Practical Plane and Solid Geometry. By Professor F. A. Bradley, London.
2. Machine Construction and Drawing. By E. Tomkins, Liverpool. Vol. I. Text, Vol. II. Plates.

3a Building Construction—Brick and Stone, &c. By R. S. Burn, C.E. Vol. I. Text, Vol. II. Plates.

3b Building Construction—Timber and Iron, &c. By R. S. Burn, C.E. Vol. I. Text, Vol. II. Plates.

4. Naval Architecture—Laying Off and Shipbuilding. By S. J. P. Thearle, F.R.S.N.A., London. Vol. I. Text, 2s. 6d. Vol. II. Plates, 5s.
5. Pure Mathematics. By E. Atkins, Leicester. 2 vols.
6. Theoretical Mechanics. By P. Guthrie Tait, Professor of Natural Philosophy, Edinburgh.
7. Applied Mechanics. By Professor O. Reynolds.
8. Acoustics, Light, and Heat. By William Lees, A.M.
9. Magnetism and Electricity. By F. Guthrie, B.A., Ph.D., Royal School of Mines, London. **Extra Vol., 3s.**

10a Inorganic Chemistry—Vol. I. Non-Metals. By Professor T. E. Thorpe, Ph.D., F.R.S.E., Yorkshire College of Science, Leeds.

10b Inorganic Chemistry—Vol. II. Metals. By Prof. T. E. Thorpe, Ph.D., F.R.S.E., Yorkshire College of Science, Leeds.

11. Organic Chemistry. By James Dewar, F.R.S.E., F.C.S., Lecturer on Chemistry, Edinburgh.
12. Geology. By John Young, M.D., Professor of Natural History, Glasgow University.
13. Mineralogy. By J. H. Collins, F.G.S., Falmouth.
14. Animal Physiology. By J. Cleland, M.D., F.R.S., Professor of Anatomy and Physiology, Galway.
15. Zoology. By E. Ray Lankester, M.A. (Oxon.), London.
16. Vegetable Anatomy and Physiology. By J. H. Balfour, M.D., Edinburgh University.
17. Systematic and Economic Botany. By J. H. Balfour, M.D., Edinburgh University.

19a Metallurgy—Vol. I., Fuel, Iron, Steel, Tin, Antimony, Arsenic, Bismuth, and Platinum. By W. H. Greenwood, A.R.S.M.

19b Metallurgy—Vol. II., Copper, Lead, Zinc, Mercury, Silver, Gold, Nickel, Cobalt, and Aluminium. By W. H. Greenwood, A.R.S.M.

20. Navigation. By Henry Evers, LL.D., Plymouth.
21. Nautical Astronomy. By Henry Evers, LL.D.
22. Steam and the Steam Engine—Land, Marine, and Locomotive. By H. Evers, LL.D., Plymouth.
23. Physical Geography. By John Young, M.D., Professor of Natural History, Glasgow University.

London, Edinburgh, and Herriot Hill Works, Glasgow.

Collins' Elementary Science Series.

DYNAMICS;

OR,

BY THE SAME AUTHOR.

Now Ready, in Post 8vo, cloth, 1s.,

LOGARITHMIC AND TRIGONOMETRICAL TABLES,

FOR APPROXIMATE CALCULATIONS.

Will shortly be published,

HYDROSTATICS AND PNEUMATICS,

In Fcap. 8vo.

LONDON AND GLASGOW:
WILLIAM COLLINS, SONS, & COMPANY.
1877.

[All rights reserved.]

181. a. 115.

PREFACE.

THE following elementary book on Dynamics has been prepared for a special purpose. It is to take its place in a series adapted for the requirements of Science Classes, under the Science and Art Department. I have, therefore, at the request of the publishers, made the order in which the various subjects are treated agree closely with the scheme drawn up by the Science and Art Department, and issued in the Government Directory. I am in hopes, however, that the book may be found useful to others besides those for whom it is primarily arranged.

Without the assumption of more mathematical knowledge than students taking the "First Course in Theoretical Mechanics" are supposed to possess, it would be impossible to give satisfactory proofs of all the propositions that must be considered even in the elementary theory of Dynamics. This may be said, for example, of some of the propositions in the theory of Couples, of those regarding Circular Motion, and even of the Parallelogram of Forces. In such cases I have endeavoured, instead of attempting to put forward imperfect proofs, simply to give clear statements of the propositions, or of the meanings and the objects of the formulas, with explanations which may enable the student to retain the propositions or the formulas in his memory. When he reaches a more advanced stage he will be better able to enter into the mathematical demonstrations. In the meantime, he should aim at obtaining clear physical conceptions regarding the first principles of Dynamics, and should learn to express his knowledge with precision and definiteness.

I have to acknowledge the great obligations that I owe to Professor James Thomson, LL.D., for the help that he has given me, critically and otherwise, in the preparation of various parts of this book, and particularly in the writing of the chapter on Gravitation and Terrestrial Gravity, in which improved modes of exposition, tending, it is hoped, to increased clearness and correctness of thought, have been introduced.

J. T. BOTTOMLEY.

UNIVERSITY OF GLASGOW,
November, 1876.

CONTENTS.

PAGE

INTRODUCTION, ... 7

CHAPTER I.

Measurement of Time, Space, Mass, Velocity, Acceleration, Momentum, ... 9

CHAPTER II.

Force—Composition and Resolution of Forces, . . 24

CHAPTER III.

Properties of Matter, ... 53

CHAPTER IV.

Gravitation—Terrestrial Gravity, ... 61

CHAPTER V.

Equilibrium of Simple Mechanical Arrangements, . 78

CHAPTER VI.

On Falling Bodies, ... 94

CHAPTER VII.

PAGE
Curvilinear Motion, ... 108

CHAPTER VIII.

Work—Energy, ... 126

APPENDIX.

Tables for Comparison of British and Metrical Measures, ... 133

Exercises, ... 134

Index, ... 141

INTRODUCTION.

Dynamics is the science which treats of the action of *force.* The name is derived from the Greek word δυναμις (dynamis), meaning *force.* Within the last twenty years many improvements have been made in the nomenclature employed in this science. The name *Mechanics*, which properly denotes the science of machines, and was used by Newton in that sense, came for a time into use, instead of the appropriate word Dynamics, for the science which treats of force; and under that name there was a peculiar "cross-division" of the subject into Statics and Dynamics, in which the proper signification of the latter name was altogether departed from. The change to a better nomenclature has recently been made; but the two titles of this book will stand for a time together on the title-page until the proper usage has come to be more familiar; and the double title may be regarded as a kind of monument of the transition.

Dynamics, or the science which treats of force and its effects in general, is subdivided into two branches, to which the names *Statics* and *Kinetics* are given. Statics is the branch which treats of the balancing of forces, whether acting on bodies at rest or in uniform rectlinear motion. Kinetics is the part of Dynamics which treats of forces acting against inertia of matter, and producing motion, or change of motion. For example, under Statics are considered such questions as the relation between forces applied to a lever when those forces are in equilibrium. Under Kinetics fall such questions as the setting in motion of a bullet by the pressure due to the explosion of gunpowder in a gun, or as the regulating action of the fly wheel on the speed of a steam engine.

Before entering on the consideration of the effects of force, and of the measurement of force, it is necessary to obtain some fundamental notions concerning the measurement of *time*, *space*, *mass*, *velocity*, *acceleration*, and *momentum*. To these, therefore, the first chapter, which may be regarded as introductory to Dynamics, is devoted.

THEORETICAL MECHANICS.

CHAPTER I.

MEASUREMENT OF TIME, SPACE, MASS, VELOCITY, ACCELERATION, MOMENTUM.

1. **Measurement of Time.**—Our primary idea of time is got from observing the succession of events, and, in particular, the succession of our own ideas. We notice external phenomena, and we receive from them, and from the succession of ideas that they call up in our minds, the mental impression of duration. At the same time, we form involuntarily a rough comparative estimate of the length of time that elapses between one event and another. To take a simple example, looking at a ball rolling along a smooth horizontal table, or a curling stone on smooth ice, we see the body successively occupying different positions, and we observe the time that elapses during its passage from one position to another. Almost unconsciously we compare the time taken to traverse 10 feet with the time taken to traverse 20 feet; and if other events are occurring at the same time, for example, if a second ball or stone is moving after the first, we perhaps compare the time taken by the second body to pass over 10 feet with the time taken by the first to traverse the same distance. Even when there is no observation of external changes, as when we sit perfectly still in a dark room, the succession of our thoughts is accompanied by consciousness of duration; and this mental sensation, if we may call it so, of the passage of ideas through our minds, constitutes our perception of the flow of time.

The succession of our ideas does not, however, afford us any standard by which we can measure intervals of time; and still less has one individual the means of comparing the passage of ideas in his own mind with the passage of ideas in the mind of another. It is the observation of external phenomena that must be used for the measurement of time.

The phenomenon in nature which all mankind have adopted as the foundation of the measurement of time is the periodic rising and setting of the sun. By the least civilised and by the most highly cultivated, the same general standard is adopted, although in the latter case, as we shall now explain, the irregularities to which the rising and setting of the sun are subject are done away with, for the purposes of reckoning, by considering an average instead of the real motion.

2. Standard of Time.—The *standard* of time for astronomical purposes is the *Sidereal Day.* It is the time that the earth takes to revolve on its own axis; and the length of it is determined by observing the interval that elapses from the moment that a fixed star appears to cross the meridian of any place, till the same star again crosses that meridian, going in the same direction. It is by this means that clocks of astronomical observatories are set from day to day. Part of the daily work of every observatory is to determine the true time;* and it is by means of determinations made in observatories that clocks and watches, on land, are set in civilised countries. The length of a sidereal day has been shown by calculation to be very nearly constant. By means of records of ancient observations of eclipses, it appears probable that, since the year 720 B.C., the time of a revolution of the earth on its axis has become increased by not more than $\frac{1}{2,700,000}$ of its length, that is, by about $\frac{1}{31}$ of a second of time.

---

* It will be noticed that the word "time" is used in two senses in the English language. In one sense an interval of time is meant as "a long time," "a short time," the "time taken by a stone to fall 30 ft. from rest;" in the other, a particular instant in duration, as "the time of day," the "time at which exterior contact in a solar eclipse commences."

From the sidereal day the *Mean Solar Day* is easily derived; and the mean solar day is, as we have said, the *unit* by which time is measured for ordinary purposes. The mean solar day is the *average* interval of time that elapses between two passages of the *sun* across the meridian. It is about four minutes longer than the sidereal day, owing to the motion of the earth in its orbit round the sun. The mean solar day is divided into twenty-four hours; each hour is divided into 60 minutes, and each minute into 60 seconds. Besides these units there are the *Mean Solar Year*, consisting of 365·242 mean solar days, and the *Century*, consisting of 100 mean solar years. These are used for expressing the measure of long periods of time.

3. Measurement of Space.—Space is measured by arbitrarily defined units. By Act of Parliament, the primary British Unit of length is the *Imperial Yard.* It is defined to be the distance between two marks on a certain metallic bar, which is kept in the Office of the Exchequer, when the whole bar has a temperature of 60° F. The yard is not derived from any fixed length in Nature. It came gradually into general use; and when the time arrived that it was necessary, owing to advance of civilization, to settle definitely an imperial standard, the standard yard was defined in the way just stated. To preserve the standard, the relation between the length of the yard and that of a seconds pendulum, at a particular station, has been very accurately determined; and were the standard yard lost or destroyed, it could be replaced by means of pendulum measurements. It could also be replaced by means of the carefully measured base lines of the Ordnance Survey, except in the case of the alteration of these lines by earthquake disturbance. There are, however, certain authorised copies of the standard yard kept in various places, for the purpose of replacing the original standard if necessary.

The yard is divided into three feet, and the foot into twelve inches; and besides these lengths there are the mile, the furlong, the fathom, and a number of others.

The foot and inch for the measurement of short lengths, and the mile for measurement of great distances are, however, those most generally used.*

4. The Metric System.—The Metric System of length measurement was invented in France. It is founded on the *Metre*, which was originally defined to be the ten-millionth part of a quadrant of the earth's meridian, from the pole to the equator. By means of careful measurements, the length of a certain quadrant was estimated, and a standard was constructed; and it is that standard and the accurate copies of it, kept in France, in London, and elsewhere, that now practically define the metre. The metre is divided decimally, one-tenth of a metre being called a decimetre, one-tenth of a decimetre, or one-hundredth of a metre, a centimetre, and one-tenth of a centimetre, or one-thousandth part of a metre, a millimetre. One thousand metres is called a kilometre. In order to avoid the multiplication of names, however, it is convenient to use only the centimetre and the kilometre; the centimetre for the measurement of short lengths, and the kilometre for greater lengths, such, for example, as we should measure in miles.

The centimetre is chosen as the practical unit of length on account of its connection with the unit of mass on the metrical system; and, by using it, calculations regarding masses and weights are much simplified. Thus it will generally be found, in dynamical questions, more convenient to speak of 1000 centimetres than of 10 metres. The kilometre is used because the numbers required to express a few miles in centimetres would be inconveniently great.

In the Appendix will be found a table comparing the weights and measures of the British system with those of the metric system. The student will do well to accustom

---

*** Unfortunately there are two lengths, each called a mile: the statute mile, of 1760 yards, and the sea mile of 2029 yards nearly. Of these, the latter alone is convenient for geographical measurements. It is the length of one mean minute of longitude at the equator, and is thus admirably suited to measurements for geographical and nautical purposes.**

himself to the use of the latter system, not only on account of its convenience, so far as scientific calculation is concerned, but also because it is daily gaining ground in England, and even more rapidly abroad, and must shortly become universally employed.

5. Mass or Quantity of Matter.—Among the properties of matter that present themselves directly to our senses, are certain properties which lead us to consider differences in *quantity of matter*, both in bodies of the same material and in bodies of different materials. We find, of bodies whose composition is the same, some large and others small; and everything else being the same, we consider of course that, of two bodies, the one which has greater size or volume contains the greater quantity of material. Again, we meet with some bodies that are close-grained and compact, and others that are loosely arranged and perhaps porous, and though it would be fallacious to conclude that a body whose pores are visible or easily perceptible necessarily contains but a small quantity of matter, nevertheless we are led to the notion of a larger or smaller quantity of material being packed together in a given space. Again, when we think of the *weight* of bodies, we find, for the most part, that compact and close-built bodies are heavy in comparison with the loose and porous ones. Lead, for example, weighs much heavier, bulk for bulk, than wood or chalk, and thus we naturally conclude that the weight of bodies depends to some extent on their mass, finding, as we do, that for bodies of the *same* material double bulk has double weight. Lastly, when we consider the effect of forces in setting different bodies in motion, we perceive that the *inertia* is greatest in those that weigh heaviest, and that appear most compact. Compare, for example, the force that it would be necessary to exert in order to strike off with the same velocity a common cricket ball, a ball of worsted, and a ball of lead, the size of all the balls being the same. From inertia we thus get the impression of mass, or quantity of matter.

6. Mass, like time and length, is measured by comparison with an arbitrary standard.* The British unit of mass is the *Imperial Pound.* It is a quantity of matter equal to that of a certain piece of platinum that is kept in the Office of the Exchequer. The unit of mass is defined thus by Act of Parliament, and certified copies of the standard are kept in various places for the purpose of replacing it, should it be accidentally lost or destroyed.

The unit of mass in the metrical system is the *Gramme.* It was primarily defined to be equal to the mass of one cubic centimetre of distilled water at the temperature of maximum density of water. Practically, however, the standard unit of mass in the metrical system, as in the British system, is a certain piece of platinum defined by Statute.

DEFINITION.—Two bodies are said to be of equal mass when they have equal inertia.†

7. Comparison of Masses.—Masses are, however, compared by weighing. From the beginning of the world this has been the method of measuring out quantities of matter; and though mass might be determined otherwise, there is no other method that would be simple enough to be applicable in practice.

It is owing to this, that, in common language, mass and weight are used as synonymous. The student, however, should not fail to remark that, in the majority of cases where weighing is used, it is not heaviness but the mass that is to be determined. Nevertheless, although from the most ancient times it has been tacitly assumed that a pound of one material is as heavy as a pound of another material, it was only the experiments of Galileo and Newton that justified the assumption, and that proved the grand law of nature that *gravitation is proportional to mass.* This will be more fully explained later.

---

* It is worth noticing here that the units of time, length, and mass, are the only units that are arbitrarily chosen in the absolute system of dynamical measurement. Thus the unit of velocity is unit of length passed over in unit of time; the unit of force, that force which acting for unit of time on unit of mass gives it unit of velocity. The units of time, length, and mass, are called fundamental, and the others derived units.

† Consult Arts. 19 and 37, p. 69.

8. Velocity.—When a point, or a body, is changing its position, or moving from place to place, it may do so rapidly or slowly. *Velocity is the rate at which the point or moving body changes its position.* Primarily, the word refers to linear motion, and it is understood to mean the rate at which a body moving along a line, whether straight or curved, describes its path. The term angular velocity is, however, also used, and will be explained hereafter.

Velocity may be either *uniform* or *variable;* equal spaces may be passed over in equal times, or the spaces passed over in equal times may be unequal. Uniform velocity is measured by the space passed over in unit of time; variable velocity is measured, at any particular instant or at any particular point of the path of the moving body, by the space that *would be traversed in unit of time by the point, were it to go on moving for a unit of time with the velocity that it has at the instant or at the point of the path considered.*

The space described by a point moving with unvarying speed for a certain time, is found by multiplying together the number that expresses the velocity, and that which expresses the duration of the motion. Thus, a point moving for ten seconds, with a velocity of five feet per second, traverses 50 feet; if the time of the motion is one minute, 300 feet are described. Again, if it moves only for one second, it passes over five feet; and if for $\frac{1}{10}$ of a second, or $\frac{1}{20}$ of a second, it passes over half a foot or a quarter of a foot in the time.

Let V denote the velocity of a point moving uniformly, let T represent the time of the motion, and let S represent the space passed over in the time T, then—

$$S = V\,T.$$

This is the formula for calculation in all cases of uniform motion.

9. Conversely, velocity is measured, when uniform, by dividing the number that expresses the distance traversed by the number that expresses the time required to traverse that distance. Thus, in our example, the point is found to traverse five feet in one second, and the velocity is, therefore, five feet per second; but, again, when it is found to pass over 50 feet in ten seconds, or 300 feet in 60 seconds, by dividing 50 by 10, or 300 by 60, the number, 5, which expresses the velocity of the point in feet per second, is found; or, lastly, when it is found to describe half a foot in $\frac{1}{10}$ of a second, or a quarter of a foot in $\frac{1}{20}$ of a second, on dividing $\frac{1}{2}$ by $\frac{1}{10}$, or $\frac{1}{4}$ by $\frac{1}{20}$, the same number, 5, is found for the velocity in feet per second. These considerations are of importance, for they indicate to us the method of approximating to the measurement of the velocity at a given instant, or at a given point, in cases where the motion is not uniform.

10. When a body is moving with non-uniform velocity, we must consider, in the first place, its *average* velocity. If we require to know how far a body travels in a given time, we can calculate the distance if we know the average velocity with which it is moving. Thus, a train whose average velocity is 30 miles per hour, travels 180 miles in six hours; and it is to be noticed that the result is true, though, perhaps, the train does not move over a single mile of the whole course at the exact rate specified.

The average velocity of the body is obtained on the principle on which uniform velocity is measured, namely, of dividing the whole space passed over by the time of the motion over that space. In fact, in measuring uniform velocity, what we really do is to ascertain the average velocity over the space considered; and the only difference between the two cases is that in the case of uniform velocity, the average agrees exactly with the actual velocity over the whole space, but that for non-uniform motion this cannot be the case, since the velocity is changing from point to point.

Let us now consider the measurement of the velocity of a non-uniform motion at any particular point or time. The average velocity of the motion over the whole course, is plainly not to be taken to represent the velocity over any particular short part of the course, and much less at any particular point. We may deal with the question in the following way:—

In considering the motion of the railway train over a particular part of the line, we cannot take the average velocity of the whole motion as representing it even approximately. We must take instead the time of passing over a measured portion of the line *at the place under consideration.* We may observe, for example, the time taken to traverse a mile on each side of the place, and find the average velocity of the motion over these two miles. This will evidently be a more exact measure of the velocity at the particular place, than the general average over the whole line. The distance then is measured, and divided by the time, and the number obtained gives an average, it may be a good average, of the velocity at the place. But now, instead of taking the average over a mile on each side of the place, let the average velocity be taken over half a mile on each side, then over a tenth, or a hundredth part of a mile on each side. The length of path considered will be less and less, and the time of the motion considered will be less and less, but dividing the one by the other, the quotients obtained will plainly approximate more and more closely to the exact measure of the velocity at the point considered. Finally, let the distance considered be infinitesimal, and therefore also the duration of the motion, and we shall obtain an *exact* measure of the velocity at the given point. These considerations are worthy of careful thought. They led Newton to the invention of the fluxional, or differential, calculus.

11. Graphic Representation of Motion.—The motion of a body may be conveniently displayed in a diagram by means of a straight line drawn according to rule. The motion at any point of the course is completely specified, if the direction of the motion and the velocity are given.

Both of these may be represented by a straight line. For let the line be drawn from the point at which the motion is to be considered, and in the direction of the motion at that point, and, having chosen a convenient scale for the diagram, let a length of the line be cut off, numerically equal to the velocity of the moving body; that is, let the number of units of length in the line cut off be equal to the number of units in the velocity to be represented. The direction of this line will represent the direction of the motion, and its length will represent the velocity.

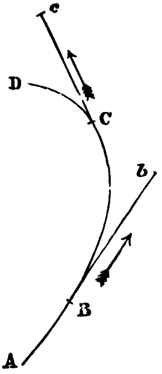

Fig. 1.

Example.—A point is moving along the curve A B C D with varying velocity. At B, its velocity is 8 feet per second, at C, 6 feet per second. To represent the motion graphically, draw from B and C the lines B*b* and C*c*, tangents to the path: it is plain that these lines indicate the *direction* of the motion at B and at C. To represent the *amount* of the velocity graphically, let us take a tenth of an inch in the diagram to correspond to a velocity of one foot per second, cut off B*b* $\frac{8}{10}$ of an inch in length, and C*c* $\frac{6}{10}$ of an inch. These lines represent by their direction, the direction, and by their length, the velocity, of the moving point when it is passing through B and through C respectively.

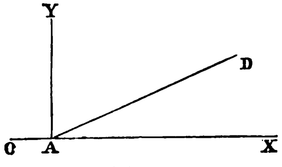

Fig. 2.

12. With the assistance of graphic representation, many useful problems with respect to motion may be readily discussed. Let a body be moving with uniform velocity along a line O X, and suddenly, as it is passing through the point A, let a velocity in the direction A Y be communicated to it. It is evident that the body will cease to move in the line O X, and will commence to move in some line such as A D. Let it be required to find the direction and velocity of the motion.

Let a ship, for instance, be moving uniformly in the direction OX (fig. 3), and at A let a man commence to climb upward at a uniform rate in the direction AY. Previous to his commencing to climb, the man was moving uniformly forward with the ship. Since the mast is being carried forward all the time that he is climbing upward, it is evident that the real motion of the man will partake both of the forward motion of the ship and of his own upward motion. We wish to consider the motion that he now has.

13. Composition of Velocities.—The finding of the actual motion of the body under two distinct velocities that are given is called the *Composition of Velocities.* The two motions separately communicated to the body are called the *Component Velocities,* and the velocity of the actual motion that results from the combination of the component velocities is called the *Resultant Velocity.*

To find the resultant of two given uniform velocities we proceed as follows:—On the line OX let us take AB to represent, in magnitude, the motion of the body along OX; and drawing the line AY in the direction of the communicated velocity cut off AC, representing its magnitude numerically. Draw also BY′ parallel to AY, and CX′ parallel to AX.

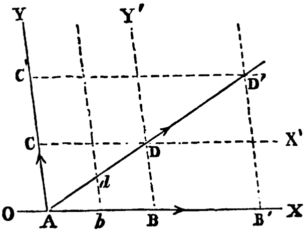

Fig. 3.

Now, taking the example of the ship, since AB represents the velocity of the ship's motion, that is, the space described in unit of time, a minute suppose; at the end of the first minute the mast AY will have been carried along to BY′; and since AC represents the distance traversed in unit of time in the upward direction, the man will have risen on the mast to the height BD (equal to AC), at the end of the first minute. The man will therefore be found at the point D. If AB′ is made equal to twice AB, and B′D′ equal to twice AC, he will evidently be found at D′ at the end of the second minute. Again, if A*b* is made equal to half AB, and *bd* equal to half AC, thus corresponding to the motions during half a minute, and if any number of points be taken in a similar way, it will be seen that he is always on the straight line A*d*DD′. Also, since AD is the line traversed during the first second, the length of that line represents, on the chosen scale, the velocity of the motion, and the number of units of length that it contains is the number of feet traversed per minute.

From these considerations we are led to the following rule for finding the resultant of two given velocities simultaneously communicated to a body. *From a chosen point draw lines in the direction of the given velocities, and cut off, on any convenient scale, lengths to represent the velocities of the component motions. On the lines thus cut off, construct a parallelogram, and draw the diagonal of the parallelogram through the chosen point. This diagonal will represent the resultant velocity both in magnitude and in direction.*

The length and direction of the diagonal may be determined by trigonometrical calculation, or by practical construction. The latter method is recommended to the beginner till he is thoroughly familiar with it.

Example.—A ball, rolling at 20 feet per second along a smooth horizontal floor, is struck at a certain point in a direction inclined at 60° to its original line. The communicated velocity is 15 feet per second, find the resultant motion. Let OX be the original line, and A the point at which the ball is struck. Take $\frac{1}{20}$ of an inch for the scale* of the diagram, cut off AB 20 twentieths, make CAB an angle of 60°, take AC 15 twentieths of an inch, complete the parallelogram, and draw AD. The angle DAB made by AD with AB gives the direction in which the ball moves from the point A. Measuring it we find it to be $24\frac{1}{2}°$ very nearly; thus the ball moves off from A in a line inclined at $24\frac{1}{2}°$ to its original line of motion. Measuring the length of AD in twentieths of an inch, we find the velocity of the ball after leaving A to be 30·4 feet per second.*

---

* The student is advised to construct a diagram for himself in accordance with the instructions in the text, or else on a larger scale, taking perhaps $\frac{1}{10}$ of an inch to represent a velocity of 1 foot per second. The angles may be laid off by means of a protractor or scale of chords.

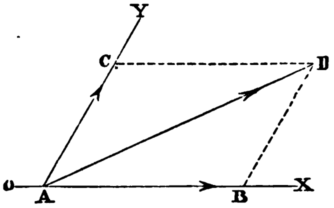

Fig. 4.

14. Resolution of Velocities is the converse of the problem of composition of velocities. The object is to find components in *given* directions whose combined motion would be equivalent to the given motion.

Example.—A body is moving with uniform velocity, 20 feet per second, up a plane inclined to the horizon at 16°. It is required to find the rate at which it is rising vertically from the horizontal plane.

Draw AX and AY horizontal and vertical. From A draw AD, making an angle of 16° with AB; cut off AD one inch long (20 twentieths). Through D draw DB and DC vertical and horizontal respectively. Measuring AC, 5·5 twentieths, we find that the body is rising from the horizontal plane at 5·5 feet per second; and if it be also required to

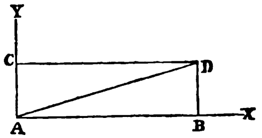

Fig. 5.

---

* Trigonometrical solution. Let P and Q represent the components and R the resultant; and let i be the inclination of the line of Q to that of P. Then $R^2 = P^2 + Q^2 + 2PQ \cos i$. Let α be the inclination of the line of R to that of P: $\cos \alpha = \frac{P + Q \cos i}{R}$.

find the rate at which it is moving in the horizontal direction, we can do so by measuring AB, when we find it to represent a velocity of 19·2 feet per second on the same scale.*

15. Acceleration.—We have considered a variable velocity, and we are now to consider the *rate* at which the velocity varies. For example, of two motions, each getting faster and faster, the velocity of the one may be increasing rapidly, that of the other but slowly. *The rate at which the velocity of a moving point increases is called its acceleration.* Thus, if a body is moving at a certain instant at 10 feet per second, and if one second later it is found moving at 11 feet per second, and another second later at 12 feet per second, the acceleration is said to be one foot per second gained per second.

Acceleration may be uniform or variable. A velocity may receive equal increments in equal times, or the increments of velocity received in equal times may be unequal. In the former case, the acceleration is measured by the quotient obtained on dividing the whole increment of velocity during any time by the time during which that increment of velocity is acquired; in the latter, just as in the case of measuring variable velocity, we consider the average acceleration; and when it is required to consider the acceleration of the moving point at any particular instant, or at any particular point in the path, we do so by considering the average acceleration for a very small space about that point, and we thus make our average agree as closely as need be with the true acceleration at the point.

The speed of the moving point may be decreasing instead of increasing: for example, when a curling stone on smooth ice gradually comes to rest. The term acceleration, however, by a convenient extension, is made to include both acceleration proper and retardation. The latter is spoken of as negative acceleration, and, algebraically, it is represented by a number or symbol preceded by a negative sign. Further, the name acceleration is very conveniently extended to include cases in which, though the speed of the motion is unaltered, the velocity is altered as to the direction.

---

* When the student understands the method thoroughly, it will be unnecessary for him to make the whole construction here indicated. The construction and measurement of the triangle DAB will evidently give all the required information. See Art. 26, Triangle of Forces.

The question of velocity does not belong, properly speaking, to dynamics. It belongs rather to a branch of *pure mathematics*, to which the name *kinematics* was given by Ampère. Kinematics considers motion itself quite irrespectively of how it may be produced. It is only when we come to speak of the production of motion by force that the laws and principles of *dynamics* are introduced. For example, we may consider equally the velocity and the path of a ball rolling on a horizontal plane, and of the shadow of some round object moving above the plane, which has the sunlight falling upon it. In the one case we may subsequently have to consider the forces that produce the motion in question, but we have, so far as the motion itself is concerned, nothing whatever to do with forces. In the latter case, we consider the laws of motion just as in the former, but here, neither directly nor indirectly, does the notion of force ever enter. In both cases, so far as mere motion is concerned, the question is purely kinematic.

As examples of motion with velocity that is varying, we shall have the cases of falling bodies, and of pendulum motion. These are the two most important cases of accelerated motion. In the one, the motion is accelerated uniformly; in the other, the law of acceleration is that of all ordinary vibratory motions.

16. Quantity of Motion, or Momentum.—The quantity of motion or momentum that a rigid body, moving without rotation, possesses is proportional to the velocity of the body; and of two bodies moving with equal velocities, that which has the greater mass has proportionally the greater momentum. The quantity of motion, therefore, if we take as unit of momentum the momentum of unit mass moving with unit velocity, is measured by the product of the numbers that express the mass and the velocity of the body. Thus, a body whose mass is 20 lbs., moving at the rate of 10 feet per second, has momentum equal to 200; and, in general, if m be the mass of the body, and v its velocity, the momentum is equal to the product mv. There is no name given to the unit of momentum or quantity of motion.

We now pass to consider the effect of forces applied to a material particle free to move.

CHAPTER II.

FORCE.

17. FORCE, the first idea of which we get from our "muscular sense," exhibits its effects, and may be observed in various ways. The tendency of force acting on a body at rest is to put the body on which it acts in motion; and it does communicate motion to the body if the body is free to move. If there are other forces acting on the body, motion may be prevented, or, as in the case of a body that is fixed, forces may be called into play as soon as a new force begins to act, which neutralise the effect of the new force. A pressure or a pull may be experienced without any motion taking place, as when the safety valve of a boiler is held down, or when a spring or cord is held stretched by the hand.

Statics is the branch of dynamics that deals with systems of force so arranged as to be in equilibrium: *Kinetics* is the branch of dynamics that deals with the motion of bodies under systems of force.

18. Accelerative Effect of a Constant Force. — A body, then, that is free to move, and acted on by a single force, will move off from rest, and it will move in the direction in which the force acts. If the force continues acting, the motion will become more and more rapid, and if, as we have supposed, the force is constant, the motion will be uniformly accelerated, equal increments of velocity being acquired in equal times. The most important case of constant, or very approximately constant force, is gravity. The force of gravity is, as we shall see hereafter, so nearly constant for places near to the earth's surface, that falling bodies may be taken as examples of motion under constant force. A stone let fall from rest moves off at first very slowly. During the first tenth of a second the velocity is very small. In one second the stone has acquired a velocity of 32 feet per second. At the end of the next second, an additional velocity of 32 feet per second has been added, so that the total velocity is then 64 feet per second. At the end of the third second, the velocity is 96 feet per second. Here the force acting is the weight or gravity of the stone.

When the body is in motion to begin with, and the force acts in the line in which the body is moving, the effect of the force in generating velocity is the same as if the body were at rest at the commencement. The velocity generated by the force is simply added to that which the body possesses, if the force acts in the direction of the original velocity, or subtracted if the force acts in the direction opposite to that of the original velocity.

For example, if a stone is projected vertically downwards from a certain point, the velocity is gradually accelerated, and increases uniformly, receiving increments at the rate of 32 feet per second of velocity gained per second; precisely as if the body had been allowed to fall from rest. In the case of a stone projected vertically upwards, the action of the force simply diminishes the velocity of projection, and 32 feet per second of velocity is lost per second, till at last the stone comes to rest at the top of the course, and then commences to descend. When the original velocity is not in the same line with that of the force, the path of the body will be curved. This case we cannot consider here.

The quantity of motion generated in a given time is proportional to the intensity of the force. If a certain constant force, acting for a second on a given mass, produces a velocity of 10 feet per second, a double force will produce a velocity of 20 feet per second, in one second, and a triple force a velocity of 30 feet. Again, remembering what has been explained as to the measurement of quantities of motion, if a given force, acting for one second, gives to a mass of one pound a velocity of 10 feet per second, a double force acting for the same time on a mass of two pounds will give it a velocity of 10 feet per second, and a triple force, an equal velocity to a mass of three pounds.* From this, and from what has been stated above, it follows that the velocity produced by a given force in a given time is directly proportional to the magnitude of the force, directly proportional to the time during which the force acts, and inversely proportional to the mass of the body. This may be expressed by a formula in the following way:—Let F be the numerical measure of the force, that is, since we have not yet explained how forces are measured, a number proportional to the magnitude of the force on any system of measurement whatever.† Let t be the time during which it acts on a mass equal to m, then v being the velocity generated—

$$v \text{ is proportional to } \frac{F}{m} \times t$$

The explanations that have just been given are summed up in the first two of Newton's celebrated *Axiomata sive Leges Motûs.*

19. Laws of Motion.—LAW I.—*Every body continues*

---

* Bodies of great mass and bodies of small mass fall equally fast, because as the mass increases, the force of gravity upon the mass increases in the same proportion.

† It may be in terms of some quite arbitrary unit;—the force exerted by an arbitrarily chosen spiral spring when pulled out *one foot,* for instance.

in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by applied forces to change that state.

Matter thus offers resistance to forces applied to change its state as to rest or motion. It is this resistance that is experienced when we attempt to open a "heavy"* door quickly, or to stop it suddenly when it is swinging on its hinges. *Inertia* is the name given to the property of matter by which it resists forces tending to change its state as to rest or motion.

LAW II.—*Change of motion is proportional to the force applied, and takes place in the direction in which the force acts.*

This law includes both the case that we have just been considering, and also that in which force acts in such a way as to alter the *direction* of motion. In the latter case, the velocity generated by the force in the line of its action compounded, according to the principles of composition of velocities explained above, with the velocity that the body already possesses, gives motion to the body in a new line. We cannot enter into this subject here in detail. A particular example of great importance will be considered under the name Uniform Circular Motion, Chap. VII.

20. Elements required for the Specification of a Force.—The elements required to specify a force, that is, the elements that must be known before we can form a clear idea of the force, and calculate its effects, are three:—(1) The place of application of the force, (2) the direction of the force, and (3) its magnitude.

1. *The place of application of a force.*—We may con-

---

* The word "heavy," though universally employed, is plainly not scientifically appropriate. The *gravity* of the door is borne entirely on the hinges, and the resistance spoken of is not due to gravity, except indirectly, in so far as friction of the hinges may add to it. In the case of stopping a door which is swinging, the gravity of the door does not affect the resistance experienced even indirectly. "Massive" would be quite appropriate, but, with our present usages, would frequently sound pedantic.

sider, first, the case of a force applied at a point. This case, though we cannot actually obtain or handle a particle of infinitely small dimensions, and have no case of a force applied in a single mathematical line, is nevertheless practically realised in many cases. For example, it is realised in the case where a heavy ball is hung up by a string; or, as in the figure, when the ball is hung up by a string, and pulled aside by another string attached to a point in it. The forces applied here by the strings are practically the same as if the ball were a mathematical *point*, endued with heaviness, and held in its position by forces applied in mathematical lines. The dimensions of the strings have, in fact, nothing to do with the question of the forces that they transmit. It is merely the property of the string by which it transmits the force in a straight line from point to point along its length that is considered.

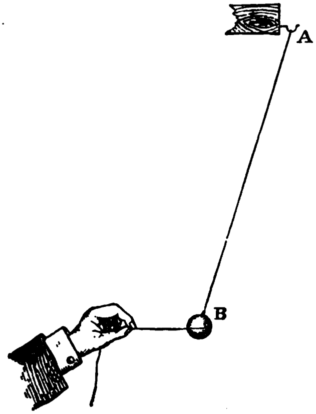

Fig. 6.

The next case is that in which force is distributed over an area. The pressure of the steam on the piston of a steam engine is an important example. Here when we consider the direction of the force, we must observe whether it is applied normally to the surface or not; and when we consider the amount of the force, we must reckon it in terms of the amount of force applied per unit of area.

Lastly, we may have a force applied throughout the whole mass of a body. Thus gravity acts on every particle of each body. By means of the connections of the particles of a rigid body, the forces on the several particles are transmitted from point to point, and it is thus that, while the force of gravity is really applied to the excessively minute particles of which a rigid body is composed, we are able to observe the heaviness of the body, taken as a whole.

2. *Direction of a Force.*—When the force is applied to a point, the direction of the force is that in which the force tends to move the point. In the case of a force applied to a surface, or distributed through a body of three dimensions, we may often find a line parallel to which all the forces, applied to the various points of the surface or solid body, act. For example, the pressure of steam on a plane surface is perpendicular to the surface at every point of it. The forces of gravity on the particles of a solid body, such as a stone, or a mass of lead, are all sensibly parallel, and in the direction indicated by the plumb-line at the place where the body is.

3. *The Magnitude of a Force.*—To specify the magnitude of a force, a unit for measuring forces is required; and, the unit being fixed on, the magnitude of the force is then stated by a numerical reference to it, as in all other cases of the measuring of quantities. We now proceed to explain the principle on which forces are measured.

21. Measurement of Forces. — For purposes where simplicity and ease of application are essential, we have a very convenient way of measuring forces by comparing them with easily recognised effects of the force of gravity on matter. The force to be measured, or to be specified, is compared with the *weight*, or as it is better called *gravity*,* of some known mass. For example, to measure the tension of a string, we may determine the number of pounds whose gravity would correspond to the tension of the string; that is, the number of pounds which, if suspended by the string, would, by their gravity, give a pull to the string equal to its actual tension. The pressure of steam in a boiler is always reckoned in pounds pressure per square inch; that is, in terms of the number of pounds force per square inch which it would be necessary to apply to the interior surface of the boiler in order to give it the pressure that it has, or the number of pounds with which it would be necessary to load a movable horizontal square inch at the top of the boiler in order to keep it in its place against the steam. The gravity of a ton may also be made a unit of measurement; for example, for the measurement of water pressure against a lock gate; and, on the metrical system, the gravity of a gramme, or kilogramme, or French ton, may be employed. In some of our statical illustrations we shall make use of these units. The system that makes the unit of force to be the weight or gravity of unit of mass is called the *gravitation system.*

---

* The word weight is unfortunately ambiguous, so much so, indeed, as to lead to all but universal confusion of ideas. To avoid ambiguity, the word *mass* is used as far as possible for expressing one of the meanings attributed to the word weight, viz., quantity of matter (see Art. 5), and the word *gravity* for the other, viz., the downward pressure, due to the gravitation, of the mass.

This method of measuring forces is, from its ease of application, eminently convenient in practice; and, for stating results of observations or experiments, it is easily understood, owing to our daily experience of the gravity of bodies by our sense of force. It must, however, be understood and carefully borne in mind that, in all cases where exactness and scientific accuracy are required or aimed at, it is necessary in making experiments, and in stating results in terms of the gravitation unit—that is to say, the gravity of a pound or of a gramme—to know also, and to state, the latitude of the place to which the gravitation unit that is used applies. Thus it is not sufficient to give the measure of the elasticity of a wire in pounds force per square inch (see Art. 34); the statement must be given in pounds force per square inch *at sea-level at Glasgow*, or *at sea-level in latitude* 45°, or at some other definite place. We shall see, in fact (Art. 36), that the *force of* gravity on a given mass is not invariable. It increases as we go northward or southward from the equator; and if a spring balance, graduated in such a way that at the equator the gravity of a standard 1 lb. weight would draw the index down to the mark 1, and 100 lbs. down to the mark 100, were carried to either pole, the gravity of the same masses would draw the index down to the marks for 1·005 and 100½ respectively. Thus, in going from the equator to either pole, this unit of force, the gravity of a pound or of a gramme, increases by one-half per cent.; and though such a variation as this is not of practical consequence in the measurement of forces, like the pressure on a steam boiler, it must be taken into account for philosophical purposes, and not only so, but for every practical purpose also where our knowledge is exact to a degree comparable with one-half per cent.

A clear comprehension of this subject is very important. Want of exactness in ideas as to the principles of measurement of forces cannot fail to lead to confusion. A careful comparison of the gravitation method, with the absolute system to be now described, will assist the student in understanding this subject thoroughly.

22. The Absolute Unit of Force.—The absolute unit of force, or as it is better named the *kinetic* unit of force, was introduced by Gauss. He defined the absolute unit of force to be that force which, acting for unit of time, on unit of mass, generates unit of velocity. In all kinetic questions this system is by far the most convenient; and, if any other system of measurement is adopted, for example the gravitation system, it is necessary practically, in considering any kinetic question, to go through the process of a preliminary reduction from one system to the other. Art. 55.

The definition that has just been given is a general one, and is particularised by referring to fixed units of mass, and space, and time. Thus, the British kinetic unit of force is defined to be, that force which, applied to one pound of matter for one second, generates in it a velocity of one foot per second. On the metrical system, again, the kinetic unit is that force which, acting on one gramme for one second, generates in it a velocity of one centimetre per second. We may now compare the gravitation units with the absolute units of the two systems. The force of gravity at Glasgow on a pound mass communicates to the pound left free to fall, 32·2 feet per second of velocity, in each second during which it acts; and, consequently, the gravity of a pound mass at Glasgow is 32·2 British kinetic units of force. If the pound mass is supported so as not to be allowed to fall, the force of gravity on it causes it to act downwards on its support with the force of 32·2 of these units. A gramme falling freely for one second at Glasgow, acquires a velocity of 981·4 centimetres per second; that is, the gravity of a gramme, acting on a gramme for one second, generates a velocity of 981·4 centimetres per second; consequently, the gravity of one gramme is 981·4 of the metrical absolute units above defined. Thus, *to reduce pounds force at the latitude of Glasgow* to British kinetic units, we multiply by* 32·2; *to reduce British kinetic units to pounds force, divide by* 32·2; *to reduce grammes force to metrical kinetic units, multiply by* 981·4; *to reduce metrical kinetic units to grammes force, divide by* 981·4. From what was said above it will be understood that a slightly different factor for reduction must be used at latitudes different from that of Glasgow, where the acceleration due to gravity is, very approximately, 32·2. It decreases as we go southward, and increases as we go northward. See Art. 37.

An inconvenience that has been much felt in connection with the use of this simple and beautiful system is the want of *names* for the kinetic units. This has now been done away with. A committee of the British Association has lately given the name *Dyne* to the absolute unit on the metrical system. For the British kinetic unit the name *Poundal*, indicating that the pound is the unit of mass employed, has been introduced by Professor James Thomson.

---

* Or London (see Art. 37).

23. Graphical Representation of Forces.—In discussing problems regarding forces, it is often of great convenience to represent them graphically; and, in complicated problems, such for example as the stresses in roofs or bridges, "diagrams of forces" are extremely useful. A force is specified by stating its point of application, its direction, and its magnitude; and it will at once be perceived that these are three properties belonging to a straight line. A straight line may be made to represent a force completely, thus: first, the point of application of the force may be represented by one of the extremities of the straight line; next, the direction of the force may be represented by the direction in which the line is drawn; and, thirdly, the magnitude of the force may be represented by the length of the line. Let the line be drawn in the direction of the force, and let the length of the line in units of length be made equal to the magnitude of the force in units of force; such a line represents the force completely. Thus, the diagram represents two forces applied to the point A, in directions, one at right angles to the other, and, in magnitude, that represented by A B double of that represented by A C. The arrow head placed on the lines A B and A C indicates that the forces act from A to B, and from A to C, and not from B to A or C to A. In drawing a diagram of forces it is necessary to choose the scale on which the diagram is to be constructed. Thus, where the gravity of a pound is the unit of force, if $\frac{1}{10}$th of an inch be the unit of length, a line one inch long represents a force of 10 lbs. just as in the diagram of velocities. We now proceed to consider the composition of forces.

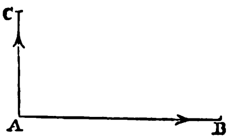

Fig. 7.

24. Composition of Forces.—When two or more forces are applied to a single point, it is always possible to find a single force which would, if it were applied to the point, have the same effect as the original forces together. Such a force is called the *Resultant* of the original system; and the forces of the original system, when considered with reference to this single force, equivalent to them, are called the *Components.* The process of finding the resultant of a system of forces is called *the Composition of Forces;* the converse process of finding a number of components equivalent to a given force is called *the Resolution of Forces.*

When there is a system of forces acting on a single particle, and not in equilibrium, we can always find a single force which will equilibrate the system. Such a force is called the *Equilibrant* of the system. It is evident that *the equilibrant is the force equal and opposite to the resultant.*

The simplest case of composition of forces that we can consider, is the case where two or more forces are applied to a point, all in the same line, and acting in the same direction. It is evident that the resultant of such a system is simply the sum of the forces. For example, if there are several men all pulling at the same rope, the pull communicated by the rope to the object that it is attached to is equal in amount to the sum of the forces thus applied to it, the rope being straight; and it is in the same line as that of the component forces, namely, in the line of the rope. Next, if all the forces applied to the point act in one line, but some in one direction and some in the opposite direction, the resultant is equal to the difference between the sum of the forces in one direction and the sum of the forces in the opposite direction, and the direction in which it acts is that of the greater of the two sums of forces. Thus, in the first case, let there be applied to the point A forces of $P = 10$ lbs., and $P' = 15$ lbs., and $P'' = 20$ lbs., all acting in the same line, AB, and in the *same* direction; the resultant is equal to $P + P' + P''$; or,

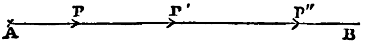

Fig. 8.

putting in numbers for P, P′, P″, to 45 lbs., and it acts towards B. As an example of the second case, P and P′, equal respectively to pressures of 10 and 15 pounds, are applied to A in the direction A B; while P″ (= 20 lbs.) is applied to A in the same line, but acting towards C. The resultant is equal to P + P′ − P″, or, putting in numbers, to 5 lbs., and acts in the same direction with P and P′, that is, from A to B.

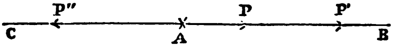

Fig. 9.

When all the forces are acting in the same direction we obtain, as we have seen, a resultant which is numerically equal to the sum of the forces, and which acts in the same direction as the components. But in the second case, when some of the forces act in one direction, and others in the opposite direction, it may happen that, on subtracting the number that expresses the sum of the forces in one direction from that of those which act in the opposite direction, the result may be zero. In that case there is no resultant; or, in other words, the forces acting in one direction *equilibrate* those that act in the opposite direction.

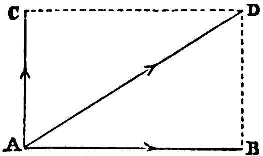

Fig. 10.

25. Parallelogram of Forces.—The next case to be considered is that in which two forces are applied to a point in different lines. Thus, let A be a point at which two forces, P and Q, acting in the lines A B and A C are applied; and let it be required to find their resultant. The resultant is found by means of the *Parallelogram of Forces.* The following is the statement of this proposition, one of the most important in the whole science of dynamics. Let the forces be represented by straight lines in the way that has just been explained. Thus, the direction of the line A B is the direction of the force P, and that of A C is the direction of Q; while the length of the line A B bears the same ratio to the length of the line A C that the magnitude of the force P bears to that of the force Q. Then, *if on the lines* A B *and* A C *a parallelogram,* A B D C, *is constructed, and the diagonal,* A D, *drawn,* A D *will represent, in direction and in magnitude, the resultant of the two forces* P *and* Q. The proposition of the parallelogram of forces may be proved in various ways. More than one strict mathematical demonstration of it, based on the assumption of certain simple axioms, have been given. It may also be proved experimentally in the following way:—

Let H and K be two pulleys, carefully mounted so as to avoid friction as far as possible, and fixed to a vertical board. Take three cords of fine silk, so as to have perfect, or nearly perfect, flexibility, and let them be knotted together at O. Let two of them, passing over the pulleys A and B, carry the weights P and Q, and the third the weight R, as in the diagram. Let the system be abandoned to itself; it will take up a certain position of equilibrium; and let us examine the relations between the forces concerned and their directions. It will be observed that the force due to the gravity of R, acting in the line OR, equilibrates the resultant of the forces P and Q in the lines OH and OK. On OH and OK take lengths OA and OB equal numerically to the forces P and Q; and construct the parallelogram OADB. Draw the diagonal OD. It will be found, on trial, that the direction of OD is *in the same straight line with the line* OR, *and that the length of* OD *is numerically equal to* R. A force, therefore, represented by OD would equilibrate the force R; and thus, since R is actually equilibrated by P and Q, OD evidently represents the resultant of the forces P and Q.

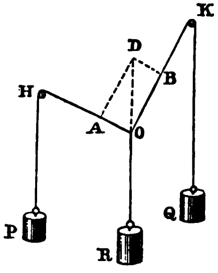

Fig. 11.

By means of the parallelogram of forces, we can always determine the magnitude and the direction of the resultant of two forces with the help of a diagram, just as with the parallelogram of velocities. Thus, if it be required to find the magnitude and direction of two forces of 25 lbs. and 17 lbs. respectively, applied to a point at an angle of 36°, we proceed as follows:—Let us take the scale of our diagram, so that a force of one pound is represented by one-twentieth of an inch. Then, drawing by the ordinary methods from a point A two lines AB, AC, inclined to each other at 36°, cut off on one of the lines AB twenty-five twentieths of an inch, and on the other AC seventeen-twentieths of an inch. On these lines construct the parallelogram ABDC. Draw AD, and measure it; measure also the angle DAB. We find that the magnitude of the resultant is 40·1 lbs., and the angle made by the direction of AD with the line AB is 14½°. Such diagrams are constructed with great ease, and give results with very considerable accuracy. The solution of problems by means of them, may be performed very quickly, and the student is recommended to make himself very familiar with their use. A divided scale, such as a footrule divided to tenths of an inch, a very simple protractor, or a scale of chords, and a pair of compasses, are all that is required for constructing the figures.

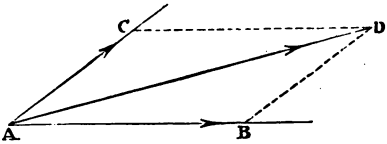

Fig. 12.

To calculate algebraically the magnitude of the resultant of two forces applied to a point, trigonometrical formulas* are generally required. In one or two cases, however, the calculation is simple. Thus, in the case where the directions of the components are at right angles (see fig. 10), the parallelogram becomes a rectangle. In that case the square of AD is equal to the sum of the squares of AB and BD; or the magnitude of the resultant of the two forces P and Q is equal to $\sqrt{(P^2 + Q^2)}$. The case where the components are equal, and one inclined at 120° to the other is equally simple. There (fig. 13) the resultant is equal in magnitude to either of the components, and its direction bisects the angle between them.

---

* See note, Art. 13, where the trigonometrical solution is given.

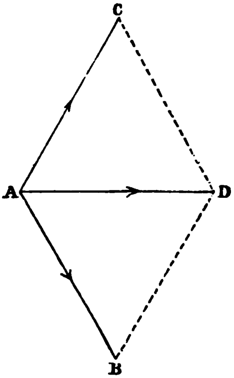

Fig. 13.

When there are more than two forces applied to a point, and it is required to find their resultant, all that is necessary to do is to apply the parallelogram of forces, so as to take in successively one of the components after another. Thus let three forces P_1, P_2, P_3, be applied to A, and let it be required to find their resultant in magnitude and in direction. First, construct on the lines that represent P_1 and P_2 a parallelogram, as is done with the dotted lines in the diagram, and draw the diagonal R′. As we have seen, this is the resultant of P_1 and P_2. Being their resultant, it may be used instead of them; and, in fact, if P_1 and P_2 are removed from the system, and their resultant substituted for them, the system will be unchanged. Now construct a parallelogram on this line R′ and the line that represents P_3, and draw the diagonal. It is plain that it represents in magnitude and in direction the resultant of the three forces P_1, P_2, and P_3. If there were a greater number of forces, the application of the parallelogram of forces a sufficient number of times would at last give us the resultant of the system.

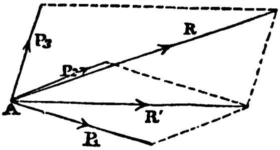

Fig. 14.

26. Resolution of Forces.—When a given force is to be *resolved* into components in two given directions the method illustrated in Art. 14 is to be employed. Let a line AD (fig. 12) be drawn representing the given force in magnitude and direction. Let AB and AC be the given directions in which it is required to find two components equivalent to AD. Drawing the parallels DB and DC, the lines AB and AC thus cut off evidently represent in magnitude the components of AD in the given directions.

27. The Triangle of Forces.—Let AB and AC represent two forces, P and Q, applied to a point whose resultant or equilibrant it is required to find. We construct the parallelogram ABDC, and draw the diagonal. The line AD is the resultant acting from A to D. Consequently also DA, representing a force equal to the resultant, but acting from D to A, is the equilibrant. Now, if instead of constructing the parallelogram, we simply draw from the point A′ a line A′B′, representing the force P in magnitude and in direction, and from the point B′, the extremity of the line thus drawn, draw B′D′ representing the force Q in magnitude and in direction, then it will be seen, on comparing the equal and similar triangles ABD and A′B′D′, that the line D′A′, which closes the triangle, represents, when taken going round the triangle in the same order with A′B′ and B′D′, the equilibrant of the forces P and Q; and, taken in the reverse order, that is, from A′ to D′, it represents their resultant.[*]

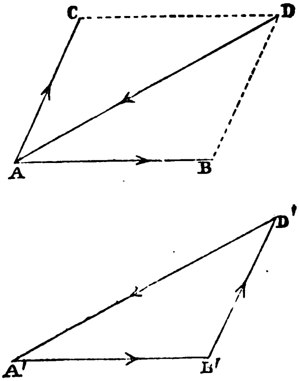

Fig. 15.

---

[*] It must be particularly noticed that the point of application of the forces is not here represented. It is only the relative magnitudes and directions that we are concerned with.

This is the principle of the *Triangle of Forces.* The statement of the principle constitutes a proposition which, in many cases, simplifies the constructions necessary to find the resultant or equilibrant of a given system of forces. The statement of the principle is as follows:—If two forces applied to a point are represented in magnitude and in direction by two sides of a triangle taken in order, the third side of the triangle taken in the same order will represent in magnitude and in direction the equilibrant of those forces, and taken in reverse order will represent their resultant. The principle may also be expressed as follows:—

If three forces, when applied to a point, are in equilibrium, they can be represented by three sides of a triangle taken in order; and, conversely, *if three forces are represented in magnitude and in direction, by the sides of a triangle taken in order, they will be in equilibrium if applied to a point.*

27A. The Polygon of Forces.—Let there be any number of forces, P_1, P_2, P_3, etc., applied to a point, and let it be required to find their resultant. This, as we have seen (Art. 25), may be done, by applying over and over again the construction for the parallelogram of forces. The method indicated may be carried out numerically, or by the method of practical construction, without any difficulty.

The resultant of any number of forces applied to a point may, however, be obtained by means of the *polygon of forces.* Let any number of forces be applied to a point, then if they can be represented in magnitude and in direction by the sides of a closed polygon taken in order, they are in equilibrium; and, if they are in equilibrium, they can be represented by the sides of a closed polygon. But if they are not in equilibrium, then, drawing in order, as in the triangle of forces, lines to represent them in magnitude and in direction, it will be found that the polygon thus formed will not be closed. In that case the line that closes the polygon, taken in the same order with those of the component forces, represents in magnitude and in direction the equilibrant of the system, and taken in reverse order represents the resultant of the system. These propositions will be easily proved by the application either of the parallelogram of forces, or of the triangle of forces, so as to combine successively the forces of the given system.

The diagram shows the two methods of finding the equilibrant E of four forces applied to a point. The student will have no difficulty in understanding it on comparing it with the diagram for the triangle of forces, Art. 26.

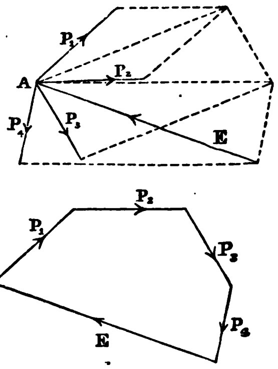

Fig. 16.

28. Parallel Forces.—We next consider the composition of two parallel forces applied to a rigid body; and, in the first place, that of two parallel forces acting in similar directions. Except in a single case, which will be considered hereafter, it is always possible to find the resultant of two parallel forces, that is, the force which, if applied to the body to which those forces are applied, would be equivalent to them; or, on the other hand, to find a force, the equal and opposite of the resultant, which shall equilibrate them. It is, of course, by means of the rigid connections of the body to which the forces are applied that the effect of the forces is transmitted, and that a single force applied to the rigid body is able to counterbalance two or more forces.

Let there be two forces, P and Q (fig. 17), applied at the points A and B, the directions of P and Q being similar, it is required to find the resultant of P and Q. Join AB. *The resultant of two parallel and similar forces,* P *and* Q, *is a force equal to the sum of* P *and* Q. *It acts in a line* *parallel to the lines of* P *and* Q, *and its line of action cuts the line* AB *in a point* C, *such that* AC : BC :: Q : P, or, as it is commonly said, *the point* C *divides* AB *into parts inversely proportional to the magnitudes of the forces.* This is the principle of the lever. The arms of the lever are inversely proportional to the forces applied to them if the forces are parallel.

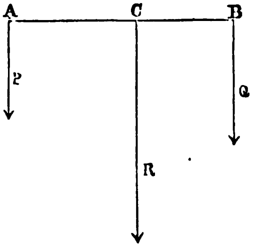

Fig. 17.

An experimental proof of the proposition may be given in the following way: Let A B be a very light but strong bar, and let two weights, P and Q, be suspended from its extremities. Let a silk cord, capable of sliding along the bar A B pass over the pulley O, and have weights attached to its extremity. It will be found on trial that there is equilibrium when the silk cord is at the point C, which divides A B inversely as P to Q; and that the weight R, required to counterbalance P and Q, is equal to P + Q, the sum of the components. Thus the proposition is established.

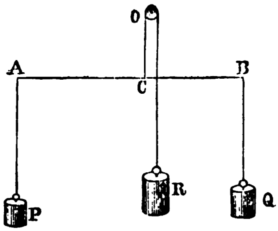

Fig. 18.

The experimental proof is only interfered with by the weight of the bar A B. This may be made very small in comparison with P, Q, and R. The effect of it may, however, be got rid of altogether by substituting for the bar, A B, a board placed on small glass balls or marbles, and, with the aid of pulleys, applying the forces P, Q, and R, horizontally. Other forms of experimental demonstration may be given; and there are, besides, geometrical methods of proving the proposition.

Taking a numerical example, let us suppose that the light bar, in the figure, is 24 inches long, and that the weight P, attached to the extremity A, is 10 lbs., and the weight Q, attached to B, is 24 lbs.; then, according to the rule, R, which is equal to P + Q, is 34 lbs.; and the point of application, C, of the string that communicates the effect of the force R to the bar A B, is situated so that

$$10 : 24 :: BC : AC.$$

From this proportion we get at once

$$10+24 : 10 :: BC+AC : BC,$$
$$\text{or } BC = AB \times \tfrac{10}{34} = \tfrac{5}{17}\,AB.$$

But, as A B is 24 inches, we find that B C = 7·06 inches, nearly.

In finding the resultant of P and Q we have also found the equilibrant of those forces, or the force which, if applied to the system, will keep P and Q in equilibrium. R being the resultant or equivalent of P and Q, a force that will equilibrate R is evidently the equilibrant of P and Q. Thus R′, the equilibrant, which is represented by the dotted line in the diagram, is equal to the sum P + Q; it is parallel to their lines, but dissimilar in direction, and it cuts the line AB, in C, inversely as the forces P and Q.

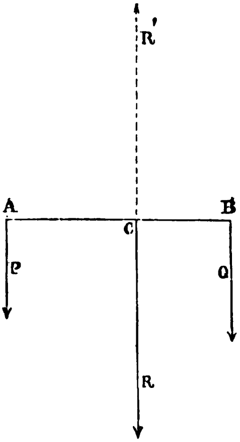

Fig. 19.

When the two components, though parallel, are in dissimilar directions, and when they are unequal in magnitude, the method of finding the resultant is as follows: Let P and Q (fig. 20) be applied in lines passing through A and B, and let Q be the greater of the two. Through B, the point at which the greater of the components is applied, produce the line A B to C, such that

$$AB : BC :: Q-P : P.$$

The resultant of the forces P and Q is a force equal to the difference Q − P, and it is to be applied at the point C found as above. Thus *the resultant of two unequal, dissimilar, parallel forces is equal to their difference. It is parallel to their direction, and it acts in a direction similar to that of the greater component. Its line of action meets any line, drawn across their lines and produced through the point in which it cuts the greater, in a point such that the segment of the line between their lines is to the produced part as the difference of the two forces is to the smaller of them.* Thus A B is a line 12 inches long; the force applied at A is 6 lbs., that at B, 10 lbs., and the forces are parallel but dissimilar. The resultant is a force of 4 lbs.; it acts in the same direction as the force 10 lbs. applied at B; and it passes through the point C in the line A B produced through B, the position of C being such that

$$AB : BC :: 4 : 6$$

Thus $$BC = \frac{6 \times 12}{4} = 18 \text{ inches.}$$

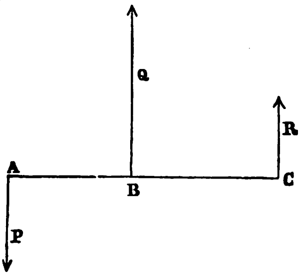

Fig. 20.

To prove this proposition it is only necessary to observe, that if we have three forces in equilibrium, any one of them may be considered as equilibrating the other two; and that, if we have three forces in equilibrium, a force equal in magnitude and opposite in direction to any one of the three, may be taken as the resultant of the other two. Now we have just seen that three parallel forces, P, Q, and R′ (fig. 21), equilibrate if Q is equal to $P+R'$, and parallel to those forces, but dissimilar in direction, and if the three forces are applied at points A, B, C, such that

$$P : R' :: BC : BA.$$

If, then, we take a force R equal and opposite to R′ this must be the resultant of the forces of which R′ is the equilibrant. Thus the proposition is proved.

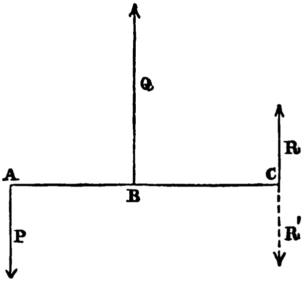

Fig. 21.

Lastly, when there are more than two parallel forces, we can find the resultant by successive applications of the propositions that have just been stated. We must find the resultant first of two of the components, then the resultant of this newly found force with the third of the components, and so on, till all the components have been introduced. The force that is found as the resultant of the last component, and of the resultant of all the components except that one, is the resultant of the system.

On carrying out this process with regard to a given system, it may be found that the forces of the system are in equilibrium. This will be the case if the resultant of all the components but one is equal and *opposite*, not merely dissimilar in direction, to the last component. It will be seen that when there is equilibrium among the forces we must have the sum of all the forces acting in one direction equal to the sum of all the forces acting in the opposite direction; or, calling those that act in one direction positive, and those that act in the opposite direction negative, we must have the algebraic sum of the components equal to zero.

When the algebraic sum is zero, we must next find, first, the line of application of the resultant R of the sets of forces acting in one direction, then the line of application of the resultant, R′, of all the forces acting in the opposite direction; then, if these lines coincide, the forces in the system will be in equilibrium. If the lines do not coincide the forces will not balance. The resultant, nevertheless, will not be a force. It will be a *couple.* Cases such as this we now proceed to consider.

29. Couples.—The case, to which we referred at the beginning of Art. 28, in which a single resultant cannot be found for two parallel forces, is that in which the forces are *equal and dissimilar in direction but not opposite.* It is evident that, in this case, the rule given above for finding the resultant of two parallel and dissimilar forces does not apply. Two forces that are equal, parallel, and dissimilar in direction form a system called a *couple.* Such a system, acting on a rigid body, cannot be equilibrated by the application of any single force to the body.

When a couple is applied to a rigid body it tends to make it rotate. It has no tendency to give it any motion of translation, but simply to make it turn round about a certain axis, whose position depends on the form of the body. The direction in which the couple tends to make the body rotate depends on the arrangement of the forces of the couple. In the figure the couple represented as (*a*), applied to a rigid body, would evidently tend to make it rotate in the direction of the hands of a watch face upward. The couple (*b*) would make it rotate in the direction opposite to that of the hands of the watch. A couple is said to be positive or negative according to the direction in which it tends to rotate any body to which it is applied. It is, of course, a matter of indifference which direction is chosen to be positive, the choice being made quite arbitrarily: but one direction having been fixed on as the positive direction, every couple that would have a similar effect is said to be positive; and every couple that would have the opposite

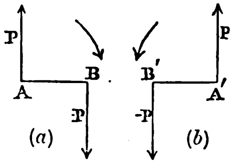

Fig. 22.

effect as regards rotation is called negative. Thus, if a couple similar to (*b*) is called positive, which is very commonly done, a couple similar to (*a*) is negative.

In the theory of couples, the perpendicular distance between the lines of the forces, that is, the line AB in the diagram, is called the *arm* of the couple. The *moment* of the couple is a number that measures its value and expresses its importance (moment) in producing rotation. It is the number found by multiplying together the number that expresses the length of the arm, and that which expresses the magnitude of the force that is acting. Thus, if P is 6 lbs., AB 3 feet, the moment of the couple is equal to 18—the pound-force and the foot being the units of force and length.*

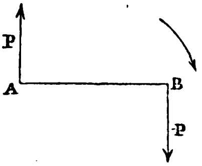

Fig. 23.

A couple cannot, as has been said above, be balanced by a single force. It can only be equilibrated by an equal and opposite couple. Two couples are said to be equal when their *moments* are equal. Thus, two opposite couples of equal moment balance each other. The following are the laws of the action of couples when in equilibrium:—

1. Two couples of equal moment and opposite in direction, applied to a rigid body, are in equilibrium when they are in the same plane, or in parallel planes.

2. If two couples act on a rigid body in planes that are not parallel, they may be equilibrated by a single couple. The equilibrating couple is found not to be in the plane of either of the original couples.

3. When, in the same plane, there is a force applied at a certain point, and also a couple, the system is found to be equivalent to a single force parallel to, and of the

---

* Other measures of moment of couples may, of course, be used, as grammes-force and centimetres, or British kinetic units of force and feet, or again kinetic metrical units (dynes) and centimetres.

same magnitude with, the original force, but applied at a different point of the plane.

4. When there is a force acting in a line that cuts, not normally, the plane of a couple, the system is equivalent to a smaller couple in a plane perpendicular to the line of the force, and a force, equal to the original force, in a line parallel to, but not coincident with, the line of the original force.

The effect of couples in producing rotation of rigid bodies we are not prepared to consider here.

30. The Moment of a Force.—Let A B be a line representing a force P. Let O be the point in which an axis, perpendicular to the plane of the paper, cuts the plane of the paper. From O let a line, O M, be drawn perpendicular to A B, and let O M be numerically equal to p. The product of P, which expresses the magnitude of the force, and p, which expresses the length of the perpendicular, is called the moment of the force P about the axis through O. Thus, *the moment of a force about a given axis is the product of the numbers which express, respectively, the magnitude of the force, and the perpendicular distance between the line of the force and the given axis.* When there are more forces than one in the same plane to be considered, the moment of the forces about *a point* in their plane is often spoken of. The moment of a force about a point is the product of the magnitude of the force, and the length of the perpendicular from the point on its direction. The moment of a force about a point, or about an axis, is said to be positive or negative according to the direction in which the force would tend to turn or carry the body to which it is applied round the point or axis; and, as in the case of couples, one direction being arbitrarily chosen as positive, the other is taken as negative. Thus, let there be a body to which the three forces P, P′, and P″, are applied in the directions shown in the diagram. P and P′, it will be seen,

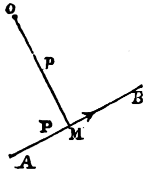

Fig. 24.

tend to turn the body round the point O, or round an axis passing through O and perpendicular to the plane of the diagram, in the direction opposite to that of the hands of a watch, face upward; while P″ tends to turn the body in the opposite direction. If the moment of P is taken as positive, that of P′ is considered as positive also, and that of P″ is negative. The positive direction is generally taken as that *contrary* to the direction of the hands of a watch, face upward.

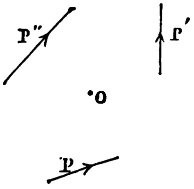

Fig. 25.

The following important proposition is commonly referred to as the "*principle of moments:*"

If there are any three forces in equilibrium, the algebraic sum of the moments of the forces about any point in their plane is equal to zero. That is to say, let there be three forces in equilibrium; and from any point in their plane let perpendiculars on their directions be drawn, and let the moments be reckoned positive or negative as explained above, then, multiplying each force by the perpendicular on its direction, and adding algebraically, that is, with the proper positive or negative sign prefixed, the sum is zero.

This proposition may be put in another form, in which it will be more convenient for us to consider it at present. Let there be two forces in the same plane, and their resultant. *The moments of the components about any point in the line of their resultant are equal and opposite.* Consider *first* two parallel forces, P and Q, and their resultant R. In the line of R take any point O, and draw O M ($=p$), O N ($=q$), perpendicular

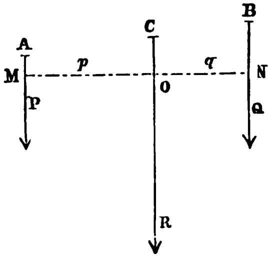

Fig. 26.

to P and Q. Since the forces are parallel, the lines O M and O N are in one straight line. Now, because R is the resultant of P and Q, the line M N is cut in O by the line of R, so that

$$\begin{aligned} P : Q &:: ON : OM \\ &:: \; q \; : \; p \end{aligned}$$

Hence, $P \times p = Q \times q$; or, the moments of P and Q about O are equal, and they are evidently opposite in direction. We have taken the case of similar parallel forces. The case of dissimilar parallel forces is left to the consideration of the student.

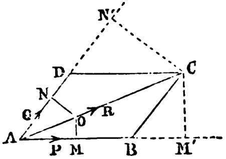

Fig. 27.

Next consider two forces, P and Q, in lines that are not parallel, and their resultant, R. Let A be the point in which the directions of P and Q meet; and let A B and A D be their lines of action. Let A C be the line of action of their resultant. From any point O in A C let the perpendiculars O M $(=p)$ and O N $(=q)$ be drawn; then the statement of the principle of moments is that

$$P p = Q q.$$

Taking A B and A D to represent the magnitudes of the forces P and Q, and constructing the parallelogram A B C D, find A C the resultant of P and Q; and from C draw C M′ and C N′ perpendicular to A B and A C. Now, by similar figures,

$$CM' : CN' :: OM : ON :: p : q.$$

But, because the rectangle under A B and C M′ is equal to the rectangle under A D and C N′, each of these being equal to the area of the parallelogram A B C D, we have the product A B × C M′ numerically equal to A D × C N′. Hence since

$$\begin{aligned} AB &: AD :: P : Q \\ \text{and } CM' &: CN' :: p : q, \end{aligned}$$

we have

$$P p = Q q;$$

or, the moments of P and Q round O equal.

The proposition that has just been proved may easily be extended to the following general statements:—

1. The algebraic sum of the moments of any number of forces in one plane, about a point in the line of their resultant, is equal to zero.

2. The algebraic sum of the moments of any number of forces in one plane, about any point in their plane, is equal to the moment of their resultant about the same point.

3. The algebraic sum of the moments of any number of forces in one plane, and in equilibrium, about any point in their plane, is equal to zero.

Numerous examples of the application of these principles will be had in the consideration of the statics of rigid bodies. For instance, the simplest way of dealing with the equilibrium, under any forces, of a lever, such as a crow-bar or a pole used for raising a heavy weight, is to consider the moments of all the forces concerned, taken round the fulcrum or point of support.

31. Centre of Parallel Forces.—It will be seen without difficulty that the method of finding the line of application of the resultant of any number of *parallel* forces given above is equally applicable whether the lines of the component forces are all in one plane* or not. The continued application of the fundamental proposition (Art. 10) taking in one component after another will finally give the resultant of all the forces. Let A, B, C, D, etc., be a number of points of a rigid body that are not all in the same plane, at which the parallel forces P_1, P_2, P_3, P_4, etc., all of which we have taken as similar in direction, for simplicity, are applied. To find their resultant we find first, the resultant, R', of P_1 and P_2, next the resultant, R'', of R' and P_3, and so on; and we finally obtain R the resultant of the whole system. R, here, is the sum of all the forces, it acts in a line parallel to the direction of the components, and its line of action passes through the point O. Now suppose that, keeping the points of application and also the magnitudes of the forces unchanged, and keeping the forces still all parallel, we could turn them all round through any angle, as is shown by the dotted lines in the diagram, it will be seen that the resultant of this system would still be a force parallel to the components, equal to their sum, and, further, that it would pass through the point O; for, the geometrical construction by which its line of action is to be found is precisely the same as before. The forces might now be turned through any other angle and still the resultant would pass through the point O.

---

* The general problem of the composition of any forces whatever, not confined to one plane, is too difficult to be taken up here.

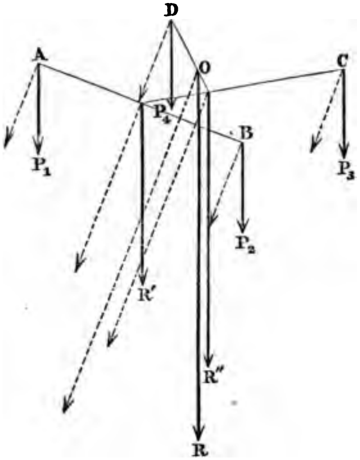

Fig. 28.

It will thus be seen that when parallel forces, given as to magnitude, are applied, not merely in given lines, but *to given points* of a rigid body, a point can be found such that the resultant of the system always passes through it, whatever be the *direction* of the parallel forces. The point thus found is called the *Centre of the Parallel Forces.*

CHAPTER III.

PROPERTIES OF MATTER.

32. We commonly recognise matter in three states, namely, the *solid*, the *liquid*, and the *gaseous*. The name *fluid* is given to either of the latter two states.

Solids are distinguished from liquids by the permanence of the resistance that they offer to forces that tend to change their shapes. A solid of given form tends to retain that form, and does not show gradual and unlimited yielding to distorting forces. Being bent or twisted it tends to return to its original configuration; and when the forces are removed it resumes that form, unless the strain has been excessive. The property of a liquid is to yield, either at once or else gradually, to forces, however small, tending to change its shape. Resistance may be experienced when the change is made rapidly, or, indeed, at any finite rate; but, if the change is made slowly enough, a liquid yields to any distorting force, however small.

According to this definition it is permanent rigidity that is the distinguishing characteristic of a solid. There are many substances which possess properties making them, in some respects, resemble a solid, and, in some respects, resemble a liquid. Such bodies are often spoken of as semi-solids or semi-fluids, or are called plastic or viscous substances. In many cases such bodies really do contain a mixture or union of solid and fluid matter. Instances of this last condition will be found in moist clay, soft mortar, sealing-wax, a bow made from a freshly cut green branch of a tree, which on being unstrung does not spring quite back to its original form unless very slowly.

Variations in degree of liquidity, or in apparent approach towards the solid from the liquid state, may be illustrated by considering such a series as the following:—first, take water or alcohol, which yield perfectly to any forces applied to them. Thus, when the finger is moved through water, the water moulds itself instantly and exactly to the form of the finger; and though, owing to the displacement of a quantity of the liquid by the finger, upward pressure is experienced, yet there is no pressure due to want of yielding of the water. Next, consider oil or tar, each of which flows but very slowly down to a level in the vessel in which it is placed. Here a resistance to change of shape is experienced; but the resistance is not permanent, for finally the oil or tar sinks down so as to take a level surface exactly. Then consider shoemaker's wax, some kinds of sealing-wax, and ice. Each of these bodies is for most practical purposes a solid, and even a brittle solid. A lump of shoemaker's wax, or a bit of ice dropped on a hard floor, breaks into a thousand fragments. Yet each of these bodies yields gradually to forces tending to change its shape. A lump of shoemaker's wax put down on a table gradually flows down and spreads itself over the table. Placed on an inclined plane, it gradually flows down the incline. The difference between wax and ordinary liquids, so far as the property that we are considering is concerned, is, that while water spreads itself over the table, perhaps, in less than a second, treacle in a minute or two, pitch, perhaps, in an hour, shoemaker's wax may take weeks or months to do the same. Ice, in glaciers, takes years to flow down a few feet of its inclined bed, though in laboratory experiments plastic yielding of ice may be shown in a few minutes.*

Considering, then, that some bodies, at first sight solid and brittle, do, when properly tested, yield, though perhaps slowly, to forces applied to change their shapes, it becomes a question whether many other bodies now called solid may not yield gradually, with extreme slowness it may be, to forces tending to change their shapes. For instance, it may be asked whether a wire of copper, or gold, or iron, hung up for a hundred years, or for a thousand, or for a million years, with a weight stretching it, will gradually yield and become elongated or not. On this subject we have no data whatever. Experiment alone can inform us, and the necessary experiments have not as yet been commenced.

---

* It is not to be supposed that the yielding of the wax and of *the* ice are due to similar properties of the two bodies.

33. Divisibility.—The ultimate constitution of matter is a subject of extreme interest and importance. It is, however, a subject regarding which we have at present but little knowledge. One of the first questions to be considered is the question of homogeneity. Are there any bodies that are perfectly continuous and homogeneous, so that dividing a body, a piece of pure metal for example, into smaller and smaller parts, all the parts, *however minute* the division, would be still exactly similar? Or, on the other hand, is there a limit reached, it may be, on dividing a cubic inch of matter millions of millions of times, such that one of the minute spaces imagined would contain a minute particle entire, or nearly entire, while another would contain a number of portions of complete particles? We have good reason for supposing the latter hypothesis to be the more probable; and the size of the ultimate particles has been estimated at from $\frac{1}{100,000,000}$ to $\frac{1}{2,000,000,000}$ of an inch in diameter.*

That exceedingly minute sub-division of matter can be actually carried on is shown by many experiments. A sheet of gold, weighing a single grain, can be beaten out into gold leaf sufficient in area to cover a square space seven inches in the side. The thickness of the gold leaf is then only $\frac{1}{250,000}$ of an inch. In this state the gold leaf is translucent, the light that passes through it being of a dark green colour. Wollaston made platinum wire of no greater diameter than $\frac{3}{100,000}$ of an inch. This was done by taking a cylinder of silver having a core of platinum, and drawing it out to the greatest possible extent. The wire thus obtained consisted, just as at the beginning, of a platinum core and a silver coating. The silver was then dissolved away with nitric acid, and the platinum wire was obtained. Experiments proving the great minuteness with which sub-division can be practically carried on, may also be made by showing how vast an area of white surface may be coloured blue with a very minute quantity of a per-salt of iron mixed with ferrocyanide of potassium to form Prussian blue; but nothing is more striking than to consider that a single grain of musk will scent a room for months or years without perceptibly losing weight.

---

* See a paper by Sir William Thomson on *Size of Atoms*, *Nature*, March 1870.

Modern naturalists have abandoned the notion of the infinitely strong, absolutely rigid, unwearable, atoms of the older speculators.

34. Elasticity. — When we attempt to change the volume of liquid or gaseous bodies, or either the volume or the form of solid bodies, force must be applied, and resistance is experienced; and, when the applied forces are removed, the body recovers its original dimensions and form exactly, except, when in the case of solids, the deformation has been carried too far. *Elasticity* is the property of matter, on account of which bodies resist forces tending to change their volumes or shapes, and by which they recover their original volumes and shapes when the distorting forces are removed.

Liquids and gases possess elasticity of volume. Solids possess elasticity both of volume and of form. Thus, when we try to compress either a liquid or a gas, resistance is experienced; and when the compressing forces are removed, the liquid or gas recovers its original volume. The same is experienced in compressing a solid, and not only so, but also when we try to bend, or twist, or stretch, the solid body; and unless the compression, bending, torsion, or elongation has been carried so far as to surpass what are called the *limits of elasticity*, the solid body recovers its volume and its shape as soon as the deforming forces are removed. Liquids and gases *offer no permanent* resistance to change of shape, and are said to have no elasticity of form, or *rigidity;* for, although when we attempt to make the change rapidly, as when a body is moved through air or through water, we do meet with resistance, yet if the change were made slowly enough, no resistance would be experienced; and, again, when the forces applied to cause the change of shape have ceased to act, there is no tendency to return to the original shape observable.

Some bodies are more compressible than others, and in the case of solids, the resistance to bending, torsion, and elongation is greater in some bodies than in others. The numerical measure of the elasticity of the body for deformation of any kind is called the *modulus of elasticity* for that kind of deformation.

In the theory of the elasticity of solids, the words *strain* and *stress* are universally employed by modern writers. A body under any kind of deformation that is resisted by elasticity is said to be *strained. Strain,* however, properly defined, *measures numerically* the amount of deformation. Thus in a wire that is stretched, the strain is the whole elongation divided by the length of the wire; or, in other words, the elongation per unit length of the wire. In a body under compression, the strain is equal to the whole diminution of volume divided by the volume of the body. For example, a wire 10 feet long (10 × 12 inches) is drawn out, by stretching, $\frac{3}{10}$ of an inch, required the strain. The strain is equal to

$$\frac{\frac{3}{10}}{10 \times 12} = \frac{1}{400}$$

or ·0025, a number which, it will be observed, is independent of the unit of length. Other kinds of elastic strain are measured similarly. *Stress* is the name given to any force or system of forces tending to produce elastic strain. Thus, we have a pulling stress which produces stretching, or compressing stresses, which may be either such pushing forces as might be applied to a bar or column to shorten it elastically, or compression applied to the solid all round, such as would be experienced were it immersed in very deep water. Then we have torsional stress, which is of the nature of a couple producing torsion in a bar or wire; and, again, we have cross-bending stresses, which consist of couples applied perpendicular to the length of a wire, rod, or bar, so as to produce cross-bending. The numerical measurement of stress depends on the nature of the stress; thus, pulling and compressing stresses may conveniently be measured in pounds force, torsional and cross-bending stresses must necessarily be measured in couples. We cannot, however, enter here into the details of measurements regarding elasticity.

The general law of elastic forces is the celebrated law of Hooke, *Ut Tensio sic Vis,* which, translated into modern phraseology, stands thus: *The Strain is proportional to the Stress that produces it.* This law is found by experiment to hold with great exactness so long as the strain is not so great that the limit of elasticity is too closely approached. When a solid mass of metal is hammered, or when a copper wire is very much stretched, as is done by bell-hangers before such a wire is put up, the metal that has been hammered is left *permanently* compressed, and the wire that has been stretched is left *permanently* elongated. Here the limits of elasticity have been surpassed, and the original state is not resumed on the removal of the compressing or stretching forces. To such cases Hooke's law does not apply; but in cases where the original form and volume are resumed on the removal of the stresses, it is generally very exactly obeyed.

The *modulus of elasticity,* for a given body or substance, is a number which expresses the resistance experienced when the body is strained with the particular kind of strain to which the modulus applies. Thus, we may have the modulus of elasticity for torsion or for stretching for a *particular* wire. Or, again, we may have to consider the modulus of elasticity for the stretching of steel wire or of copper wire in general. In the latter case we should have a number expressing the average of numerous experimental determinations of moduli of elasticity for wires of steel or of copper. In this case the modulus would be stated for a bar or wire of unit sectional area. Such a number is often called the *specific modulus* for the *substance.*

The modulus of elasticity for any particular kind of elastic strain is the number that expresses the stress that would be required to produce unit strain.* It is found experimentally by determining the stress required to produce a particular amount of strain, and dividing the stress by the strain it produces.

Example.—A piece of wire 10 feet long is drawn out $\frac{3}{10}$ of an inch by a weight of 28 lbs.; required the modulus of elasticity for that particular wire. Here the strain is

$$=\frac{1}{400};$$

the stress is 28 lbs. The modulus of elasticity for this wire is therefore

$$\frac{28}{\frac{1}{400}}$$

or 11,200 pounds force of the place of experiment (Art. 21).

Conversely, given that the modulus of elasticity for a specimen of steel wire is 11,200 pounds weight, let it be required to find the elongation produced in a wire 20 feet long by hanging on it a weight of 40 lbs. The strain is, according to the law of elasticity stated above, proportional to the stress. Let e be the total elongation in inches, then $\frac{e}{20\times 12}$ is the strain; and we have,

---

* Though the modulus of elasticity is said to be numerically equal to the stress that would produce unit strain, it must be understood that to produce unit of strain, that is, to draw one foot out to a length of two feet, would be going far beyond the limits of elasticity of any known solid. The amount of strain that can be produced without giving permanent deformation differs very much with different materials, and with the kind of strain. In some cases a strain of $\frac{1}{100}$ may be given without approaching too closely the limit of elasticity; but in many others one-tenth of this strain, or less, would produce a permanent deformation or fracture.

The student is cautioned against a common statement that the modulus for elongation or compression of a bar is numerically equal to the stress that would draw the bar out to double its length, *or would reduce its length to one half.* Such a definition is imperfect without the explanations here given; and if it be used at all the last clause should be, "*or would reduce its length to zero.*"

$$\frac{e}{20\times 12} : 1 :: 40 : 11{,}200$$

$$e=\frac{20\times 12\times 40}{11{,}200}=\frac{96}{112}=\frac{6}{7}\text{ of an inch.}$$

35. Compressibility.—All solids, liquids, and gases, are compressible; but solids and liquids, with a very few exceptions, are very slightly compressible in comparison with gases. In solids the compressibility is so small that very high pressure is required to render it sensible at all. Experiment, however, has fully established the fact that solids are compressible.

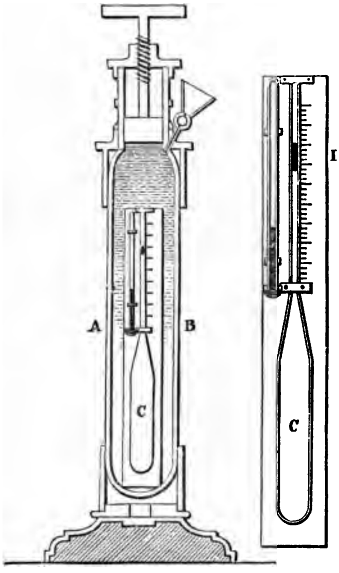

Fig. 29.

The compressibility of liquids, though also small, is easily shown by means of Œrsted's apparatus, shown in the figure. It consists of a very strong glass cylinder AB, which is filled quite full with water, and has a brass cap attached to it, in which a piston, driven by a screw, works. By screwing the piston forward very great pressure can be obtained. The liquid whose compressibility is to be tested is put into the bulb C; and a small column of mercury I, moving in the tube connected with the bulb, forms an index. The bulb containing the liquid being put into the pressure apparatus, pressure of the piston is applied, and the mercury index is seen to move down the tube. A scale attached gives the amount of diminution of volume. Observing the apparent diminution, and correcting for the *contraction* of the glass bulb itself under the pressure, the absolute diminution of volume of the liquid is obtained. The pressure is determined by means of the air-gauge shown to the left of the bulb tube.

According to recent experiments, the compressibility of water at temperature 15° C. is about 46 millionths of its volume for one atmosphere. Alcohol and ether are more compressible than water; mercury is much less so.

Gases are far more compressible than liquids. The law of their compressibility is know under the name of Boyle's law.

The resistances offered by solids to forces of stretching, bending, and torsion, are very different with different materials. Numerical determinations of elasticity have been experimentally made for most of the important metals and for some kinds of wood, chiefly for practical uses in forming structures.

CHAPTER IV.

GRAVITATION—TERRESTRIAL GRAVITY.

36. Gravitation.—Every piece of matter attracts every other piece of matter. That is to say, any two pieces of matter have a tendency to approach each other; and, in order to keep them separate, opposing forces on the two are required. *Gravitation* is the name given to this force of attraction.* In many cases this force is so small that delicate experiments are necessary to prove that it exists; but, in others, gravitation presents itself directly to our senses. Thus, the heaviness of bodies is due to gravitation. A stone held in the hand is felt to be heavy; we feel that force must be applied by the hand to keep the stone from falling. The force is transmitted through the body, and as the feet press against the earth in one direction, and the hand presses against the stone in the opposite direction, the earth and the stone are prevented from approaching each other. All the particles of the stone and of the earth are attracted or drawn towards each other; and, if we leave out of account for the present the effect of the centrifugal force of the stone, due to its daily rotation round the axis of the earth, which will be considered almost immediately, we may say that the opposite forces applied are those required to counterbalance the aggregate of all the minute forces between the particles of the stone and those of the earth. If the stone is let go, the force of gravitation sets it into motion towards the earth, and it falls downwards, or more strictly the stone and the earth fall each towards the other. Again, it is gravitational attraction that keeps the moon moving round the earth in a curved orbit. Were this force at any moment to cease acting, the moon would move on in a straight line, the tangent to its path at that instant, as shown in the figure. As it is, however, the moon, subject to attraction, moves in such a way that she describes a path round the earth which is nearly circular. Similar remarks apply to the motion of the earth round the sun.

---

* We recognise attractive forces of several different kinds. Thus, we have attraction between two dissimilarly electrified bodies, attraction between magnetised bodies, attraction between two conductors, such as two copper wires, each of which has a current of electricity flowing through it, and again we have the attraction of chemical affinity. Lastly, we have the attraction between any two masses of matter belonging without distinction to all kinds of matter. To this the name *gravitation* is given.

There has been even among the best writers on dynamics and on astronomy considerable variation in the usage of the words *gravitation, gravity, weight.* The word gravity is here taken to mean the *observed* downward force on any mass—what is sometimes called *apparent gravity.* It is the resultant of gravitational attraction, and the so-called centrifugal force as is explained in the text: and terrestrial gravity on a given mass, except at the poles, is always less than the gravitational attraction.

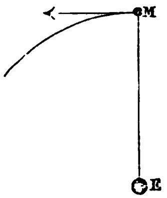

Fig. 30.

But it is not in the case of large bodies only that gravitational attraction can be observed. By a very delicate experiment Cavendish was able to observe and measure the attraction between two balls of lead.

We have good grounds for believing that gravitation belongs to all matter, not only of the solar system, but also of the various star systems that are within our view. It certainly belongs to all the matter of the solar system. The motions of the planets and their satellites, and of comets prove this; and so far as the motions of the more distant systems have been observed, there seems to be no reason for doubting that the stars that compose those systems are likewise affected by it, though we have not at present complete proof of its existence in these cases.

The laws of gravitation known to us may be stated as follows:—Every particle of matter in the universe attracts every other particle. On examining whether the gravitational attraction is more intense between some kinds of matter than between other kinds, we find that it is not so. A pound of lead attracts another pound of lead, with, so far as our experiments have told us, precisely the same force as that with which a pound of water attracts another pound of water, or with which a pound of wood attracts another pound of wood; and, again, a pound of lead attracts a pound of water with precisely the same force as that with which it attracts another pound of lead.

It is found, on the other hand, that the magnitude of the force between two portions of matter depends on their masses. If the mass of either is increased, the force is increased in the same proportion; and from this it follows obviously that if the mass of each is increased, the force is increased in proportion to the increase of the *product* of the numbers that express the masses. If, for example, we compare the attraction of 3 pounds of lead on 2 pounds of lead, with the attraction of 4 pounds on 7 pounds, the forces are as

$$3 \times 2, \text{ or } 6, \text{ to } 7 \times 4, \text{ or } 28.$$

Since the mass of a body, that is, the quantity of matter it contains, is proportional to its inertia, the law which states that gravitation does not depend on the material of which the body is composed, but is proportional to the *mass* of the body, is often briefly stated in the words:—*Gravitation is proportional to inertia.* This is a law of extreme importance with regard to the motion of bodies under gravity. Were it not so, bodies of different materials would not fall equally fast, and small and large bodies of the same material would not fall equally fast. Again, pendulums of equal length but of different materials, or of different masses, would not vibrate with the same period. It is by means of experiments on falling bodies and on pendulums that the law is proved. But, before this can be properly explained, the influence of the rotation of the earth on pendulums, and on other bodies with which we can experiment at its surface, must be considered.

Lastly, the attraction between two bodies depends on the distance between them; it is greater the smaller the distance. If the distance is reduced to one-half, the force is four times as great; if it is reduced to one-third, the force is nine times as great; if the distance is doubled, the force is diminished to one-fourth of what it was at the first distance.

The last two laws taken together give us the following statement:—*The force of gravitation between two bodies is proportional to the product of their masses, and is inversely proportional to the square of the distance between them.*

Here let it be observed, that in speaking of *the* distance between two attracting bodies, an assumption of some kind is implied. In order that the statement of the law may be really appropriate, we must either consider that the dimensions of the bodies attracting each other are so small, in comparison with the distance between the two bodies, that it is unnecessary to introduce into the question the consideration of the difference between the distance of the parts of the two bodies nearest to each other, and that of the parts farthest from each other, or else that some mean point may be taken as the distance between the two bodies. In general the former supposition is made, and the statement of the law of gravitation is, that the various *particles* of any two masses attract each other with forces proportional to the product of the masses of the particles, and inversely proportional to the square of the distance between them. The attraction, then, of any two given masses is the resultant of the attractions of the particles of those masses.

There are, however, certain cases, as will be seen immediately, in which the attraction between two bodies may be considered to be between two points. Such bodies are called *centrobaric* bodies; and these points in the bodies are called centres of gravity. For example, the attraction between two uniform spheres is the same as if the whole mass of each were concentrated at its centre. Consequently, in this case we may say that the attraction is proportional to the product of the masses of the spheres, and inversely proportional to the square of the distance between their centres. A particular case of extreme importance, in which this is very approximately true, is that of the bodies of the solar system. Much simplification is introduced into the planetary theory by our being able approximately to consider these bodies spherical, and to reckon the force between any two of them as if the whole matter of each were collected at its centre.

As an example of the law stated above, let us compare the attraction of the earth on a pound of matter at the earth's surface, and on a pound of the moon's mass. The distance of the moon from the earth is about 60 times the distance from the surface of the earth to the centre. Consequently a pound of the moon's mass is attracted by the earth with a force only $\frac{1}{(60)^2}$ or $\frac{1}{3600}$ of the force with which a pound of matter is attracted by the earth at the surface of the earth.

37. Terrestrial Gravity.—We must next consider observed terrestrial gravity, or as it is sometimes called *apparent gravity*. The observed gravity of a body is not precisely the gravitational attraction of the earth upon it. If the earth were not in motion round its own axis, then the gravity of a body would be simply the gravitational attraction of the earth on it. But bodies that we commonly speak of as at rest at the surface of the earth are not really so. Each of them is moving in a circular orbit of daily revolution round the earth's axis. Now it follows from Newton's laws of motion (Art. 19) that a body moving in a circular path with uniform velocity must be acted on constantly by a force directed to the centre of the circle in which it moves. This force has its effect in overcoming the inertia of the body, and causing the body to deviate from the rectilinear path which, in virtue of inertia, it would at any instant tend to pursue.* Thus, in order that a body at the earth's surface

---

* The student will understand this more perfectly after he has read Chap. VII. on "Curvilinear Motion." He is recommended to return to this very important subject after reading that chapter.

(a plummet hanging by its cord for example) may move in its daily circle, along with other neighbouring objects, it must receive, from without, a certain definite force directed always to the centre of the circle, and depending for amount on the mass of the body, the velocity of the motion, and the radius of the circle. Now the body is acted on by gravitational attraction, which is directed very approximately towards the centre of the earth, and this attraction resolves itself so as to supply a component which is the centreward force necessary to keep the body moving in its daily circle. The residue of the attraction, after this component has been abstracted from it, is the force which the body transmits to whatever holds or suspends it, and which, in the case of the plummet, applies itself in pulling on the string. It is this residue which we perceive by our senses as a downward force exerted by the plummet, and which we call the *gravity* of the plummet. Let P be the plummet acted on by attraction in the line PC towards the centre of the earth.* It is revolving in a daily circle whose centre is A, a point on the earth's axis NS, and the radius PA of the circle obviously depends on the latitude of

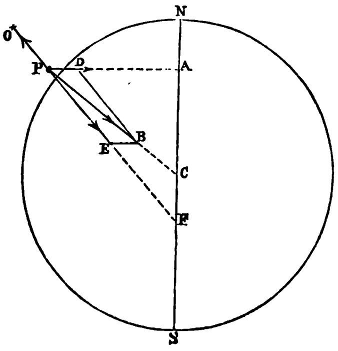

Fig. 31.

---

* Gravitational attraction is directed *very approximately* towards the centre of the earth; so nearly so, that we may, with sufficient exactitude for our present purpose, speak of it as if it were directed towards the centre exactly, and may represent it so in the diagram. Strictly, however, the attraction is not directed

the place at which the plummet is situated. Now, according to what has been said, it is requisite, in order to keep P in its circular orbit, that a definite force, though small in amount in comparison with the whole attraction, should constantly act on P in the direction PA. Let us represent the gravitational attraction in magnitude and direction by the line PB; and let us resolve it into two components, of which one, being the centreward force necessary for the daily orbital motion of the plummet, is determinate in amount and in direction, and is represented in the figure by the line PD. The other component, found by completing the parallelogram of forces, will be seen to be represented by PE. This is the force which manifests itself to us by acting on the plummet cord PO, and which is called the gravity of the plummet.

The line of the plummet cord is the line OPEF; and it will be seen from the diagram, that gravity is not, either in direction or in amount, exactly the same as the gravitational attraction on the plummet.*

The gravitational attraction of the earth on any body is, as we have seen, proportional to the mass of the body; and the force required to overcome the inertia of the body, and to cause it to move in a curved path is like-

---

precisely towards the centre of the earth, and the deviation is different at different places. This is due principally to the fact that the earth is not an exact sphere, but an oblate spheroid, being flattened at the poles and protuberant at the equator, and also that it has local irregularities, such as mountains and valleys, as well as irregularities in geological structure.

* The figure shows that this is a remarkable arrangement of forces. Though the plummet, hanging at rest by its cord, is generally thought of as being under a system of forces in equilibrium, yet it will be seen on proper consideration that the forces acting on the body are not in equilibrium; and that really the gravitational attraction on the plummet, represented by PB, and the pull of the cord on it, PO, give jointly a resultant PD, which is unbalanced, and which is the force required to keep it moving in its daily circle. This resultant goes constantly to overcome the inertia of the plummet, and to deviate it into its circular path.

wise proportional to its mass. It follows that the gravity of any body is proportional to its mass. Hence it is that all bodies, of whatever material or size, would fall equally fast in the same locality were there no resistance of the air to be overcome. This was first shown by Galileo, see Art. 50. Hence also it is that a body, whose mass is five pounds, is five times as heavy as a body whose mass is one pound. This may at first sight appear almost a truism, because the very method of showing that the *mass* of a body is five pounds is to show that it *is five times as heavy as a body whose mass is one pound.* But it must be remembered that it is only *because gravity is proportional to mass* that the method of weighing can be applied for the purpose of determining the masses of bodies.

In stating the intensity of the force of gravity on matter, it is the force per unit of mass that is given.

The force of gravity varies slightly from place to place over the earth's surface. Over the British Islands its magnitude per pound of matter is approximately 32·2 British absolute units of force (Art. 22), and this number may be taken as exact enough for most practical purposes.

As we proceed from the poles towards the equator, the force of gravity diminishes slightly. This is due to two causes. The first is the oblate figure of the earth. If the earth were a perfect sphere, and composed either of matter uniformly dense throughout, or of concentric spherical shells of uniform density, gravitational attraction would be the same at every place. But this is not the case; and, owing to the earth's figure, the attraction per pound of matter at the poles is about $\frac{1}{600}$ of its whole amount greater than at the equator. The second cause is the diminution of gravitational attraction by the abstraction of the centreward force, as has just been explained. The effect of the abstraction of the centreward force in diminishing gravitational attraction is greatest at the equator, and decreases as we approach the poles, where it is zero. This is because near the equator bodies are moving more rapidly in their diurnal circles than near the poles, the radii of the circles being greater, and also because near the equator the centreward force and the attraction are nearly in the same line, while near the poles this is not the case. The maximum diminution of gravitational attraction due to this cause is $\frac{1}{289}$ of its whole amount; and taking together the effect of the figure of the earth and of the centreward force, we find that gravity per pound at the poles is greater by about $\frac{1}{195}$ than at the equator. There are also slight variations from place to place of the magnitude and direction of gravity. Local irregularities, the presence of a great mountain or a deep ravine, or of an unusual quantity of very dense matter at a particular place, affect both the magnitude and direction of gravity; but such irregularities are insensible, except to the most refined experiments.

38. Centre of Gravity.—Let us now consider a body, A, of sensible magnitude, attracting a particle, B, placed at a short distance from it. B is attracted, as we have seen, by every particle of A. We may thus consider the whole force of attraction on B as made up of a countless number of forces between it and the particles of A; and if all of these forces are combined together into a single resultant, that resultant is the total force of attraction between A and B. The student will immediately perceive that the line of the resultant of all these forces must pass *through* the body A. Suppose now that the body A is turned round in its place, inverted, let us say. Then it is clear that the particular lines of the component forces will be altered in direction. The line B*a*, for example, will take a new direction in space. This done, we may again determine the line of the resultant of all

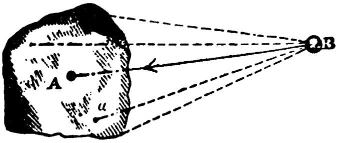

Fig. 32.

the forces between B and the particles of A in its new position. Turning A into a third position, we may again do the same. Now, there are certain cases in which, if this be done, the line of the resultant of the forces between the particle B and the particles of the body A always passes through the same point of A. When such a point can be found it is called the *centre of gravity* of the body; and bodies which possess the property of having a centre of gravity are called *centrobaric bodies.* As an example of a centrobaric body we may take a uniform sphere. Here we see at once by symmetry, that however the sphere A be turned round, the line of the resultant of all the forces of the particles of A on B passes through the centre of the sphere. The centre of the sphere is thus, according to what has been said, the centre of gravity of the sphere; and a sphere is a centrobaric body. The same applies to a uniform spherical shell.

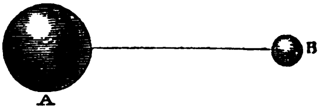

Fig. 33.

There is only a limited number of classes of bodies that possess a centre of gravity; and to find whether or not any given body is centrobaric is a problem for mathematical analysis.

The attraction between two centrobaric bodies is in the line that joins their centres of gravity. Thus, for example, we may say, that the line of attraction between two uniform spheres of matter always passes through the centres of the spheres. Newton proves geometrically that the attraction between two uniform spheres is the same as if the whole of the mass of each sphere were concentrated at its centre. This is a most important proposition in the theory of attraction.

Although, as we have seen, there are only certain configurations to which a centre of gravity belongs, yet the term is commonly used as though it had a much wider signification. Sometimes it is used instead of centre of inertia (see Art. 39); but more frequently the following meaning is attached to it: Small bodies near to the surface of the earth are affected, as we have seen, by the force of gravity. The lines in which the force acts on various parts of the body are *sensibly* parallel straight lines perpendicular to the horizontal plane at the place. The attractive forces may thus be looked upon as a system of parallel forces. The centre of this system of parallel forces being found, it is often called the centre of gravity of the body.* It must be remembered, however, that it is not rigorously true that the forces on the various points of the body are parallel; and, on the other hand, the point determined in this way for the body in one position will not be rigorously the same as the point found in a similar way for the body in another position.

Nevertheless, though the construction for finding the centre of a system of parallel forces does not in general lead to the discovery of a true centre of gravity, yet it points to an important practical operation, by which we may find a small space in any given body through which the resultant of the forces of gravity on the various particles of the body always passes. This is what is called the experimental method of finding the centre of gravity. Let us suppose for the moment that a given body is acted on by a system of forces rigorously parallel, and that, in whatever position the body is placed, the force that acts on each particle of the body remains rigorously the same. We could then find a centre of parallel forces as we have seen above (Art. 31); and we could determine the magnitude of the resultant. To find the centre of parallel forces experimentally, it would be necessary to test the body, with the forces applied, in three positions, and find in each case, by an experiment, the line in which the equilibrant of the forces acts. The intersection of the lines, determined in three properly selected positions of the body, must be the centre of parallel forces.

---

* The student must consider carefully the strict definition; and at the same time he must remember that, frequently in conversation and in examination papers, by the term centre of gravity is meant either the centre of inertia, or the centre of *parallel forces*, in the sense explained in the text.

If we now suppose that gravity acts in parallel lines, and that the position of the body tested does not affect the amount of the force of gravity on the several particles of which the body is composed, a supposition which, for practical and experimental purposes, is sensibly accurate, then the problem of finding a centre of gravity, in any case, would be simply that of finding by experiment the centre of parallel forces for the given body. A particular case will explain the method.

The figure represents a plane triangular area, that is, a thin triangular plate or lamina of uniform material, of sheet-metal for example, and it is required to find the centre of gravity of the body.

Let a fine flexible cord be attached to one of the angular points, and let the triangle hang freely by the cord. Now, it is plain that if G be the centre of all the parallel forces on the various particles of the triangle due to gravity, that is, the point through which the resultant of all those forces acts, it is necessary that G should be vertically under A, the point of support, in order that the force through A, given by the cord, may equilibrate that resultant. For if G were not in the same vertical with A, the point of support, but in such a position as G (fig. 35), the upward force through A and the downward force through G, would constitute a couple tending to bring G vertically under A. The

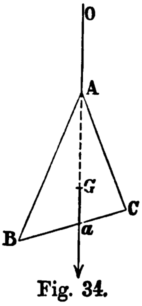

Fig. 34.

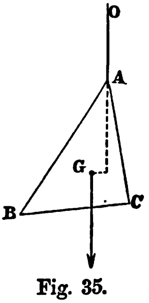

Fig. 35.

same would be true for the triangle suspended from any other point.

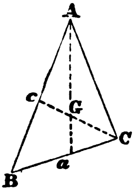

Fig. 36.

Let us now, in order to determine the centre of gravity of the triangle, suspend it by a flexible cord from the point A, and let us produce the line OA downward through the triangle to *a*. The dotted line A*a* (fig. 36) must, from what has been said, contain the centre of gravity. Suspending the triangle by the point C, and finding the line C*c* in a similar way, we perceive that C*c* must also contain the centre of gravity. Evidently, therefore, the centre of gravity of the triangle must be at G, the intersection of the two lines thus determined.

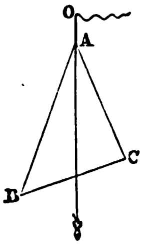

Fig. 37.

The practical way of finding the vertical line passing through O and A, and through O and C, is to suspend from the point O a small plummet, and with its assistance to mark off the line on the body whose centre of gravity is to be determined. With a little skill the method may be applied so as to lead to very accurate results.*

The method that has just been described is easily applied to the finding of the centres of gravity of plane areas. In this case, as the whole of the body lies in one plane, it is only necessary to determine two lines and their intersection, in order to find the centre of gravity.

---

* The student should experiment for himself on plane areas of various shapes cut out of cardboard. Let him support the lamina on a very fine needle passed through it (for example, let the needle be passed through A perpendicular to the plane of the triangle), and let him hang from the needle a small weight by means of a silk fibre or thread. He may then mark with pencil in each experiment *two* points through which the line of the thread passes, and drawing lines through these points determine the intersection of the lines

With bodies of three dimensions, that is, bodies having not only length and breadth but also thickness, it is of course necessary to determine three lines, and the point of intersection of those three lines. The principles to be followed are precisely the same as those that have just been illustrated. Generally, however, the form of the body interferes with the actual carrying out of the experiment. We cannot arrange to allow a plumb line to hang down through the substance of a solid mass; and it is only in the rare case where we are required to find the centre of gravity of an open framework that the method explained above can be carried out practically. In certain special cases, however, special methods may be adopted; and in every such case the method is either that which has just been explained, or one practically equivalent to it.

39. Centre of Inertia.—Having explained the term centre of gravity, and having shown how to determine it in certain cases, let us now consider the *centre of inertia* of any material system. This point has an exact geometrical definition, and can be found in any given system. The determination of it is of extreme importance in the theory of the motion of rigid bodies. In elementary dynamics it is also of great importance, because the centre of inertia is also the centre of gravity of any body that possesses a centre of gravity; and where that is not the case, the centre of inertia is the point that is approximated to in the experiment for finding a centre of gravity. Thus in bodies that possess a centre of gravity, the forces of gravity on the particles of the body balance exactly round the centre of inertia; and in bodies that have not a true centre of gravity, the forces of gravity on the several particles approximately balance round the centre of inertia.

Let us consider, in the first place, two material particles; the centre of inertia of the two particles is found in the following way:—Join the particles by a straight line, and cut the line inversely as the masses of the particles. Thus, let there be two particles, A and B, of which the masses are 10 lbs. and 15 lbs., and let the length of the line AB be 5 inches; divide the line AB in the point I, so that

$$AI : BI :: 15 : 10.$$

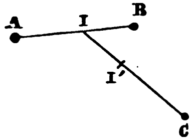

Fig. 38.

The distance of I from A is thus 3 inches, and from B 2 inches. If there be a third particle, C, whose mass is 20 lbs., then to find the centre of inertia of the three particles, find first the centre of inertia for two of the particles by the rule given above, then join that point with the third particle, and cut the joining line inversely as the sum of masses of the first and second particles to the mass of the third. Thus, find first the point I, the centre of inertia of the particles A and B; join IC, and cut it in I′, so that

$$\begin{aligned} II' : CI' &:: 20 : 25 \\ &:: \ 4 : \ 5. \end{aligned}$$

If there were a fourth particle, we should join it with I′, and cut the joining line inversely as the sum of the first three masses to the fourth; and so we should proceed for any number of particles. According to this definition, the centre of inertia is seen to be a definite point which is found by a geometrical construction. It is not difficult to show that the construction leads to the same point, whatever be the order in which the particles are taken in the process.

Any body whatever may be considered as being made up of a number of particles rigidly connected together; and when it is required to find the centre of inertia, the definition that we have given above may be applied. Geometry, however, often affords methods by which the process may be shortened, or the problem systematically attacked; and, in particular, co-ordinate geometry, with the aid of a somewhat different definition, easily deduced from that which has just been given, makes the determination of the centre of inertia for any given body simple. We may here consider a few elementary cases.

(1.) To find the centre of inertia of *a material straight line (a rod or bar) of uniform material.* Divide the line into an even number of equal minute portions. It is evident that the centre of inertia of each pair of particles, taking together the particles on each side at equal distances from the middle as a pair, is at the middle point of the line. Thus the centre of inertia of the whole line is at its middle point.

(2.) *A material parallelogram of uniform material (a thin uniform board of that shape).* Divide the parallelogram ABCD into thin bars or lines parallel to AB or CD. The centre of inertia of each of these bars is at its middle point; the centre of inertia of the whole must therefore lie in the straight line *ab*, which contains all the centres of inertia of the elementary bars. For a similar reason it must lie in the line *cd*, which contains the centres of inertia of all the elementary bars that the parallelogram may be divided into by lines parallel to AD. Thus the centre of inertia of the whole parallelogram is at the intersection of *ab* and *cd*, and it may be shown that this point is also the point of intersection of the diagonals of the parallelogram.

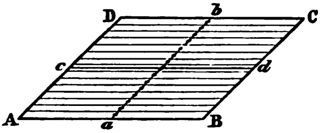

Fig. 39.

(3.) *A uniform triangle (uniform triangular board).* Let the triangle be divided up into strips or bars by lines parallel first to one of the sides BC, and then to another AC. Let the line A*a* be drawn bisecting the base BC in *a*. Then, since the line A*a* bisects each of the strips into which the triangle has been divided by lines parallel to

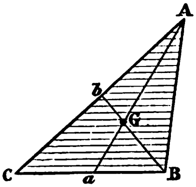

Fig. 40.

BC, the centre of inertia of the triangle must lie in the line A*a*. Drawing B*b* to bisect the side AC, it is plain that the centre of inertia must also lie in the line B*b*. Consequently G, the point of intersection of A*a* and B*b*, is the centre of inertia.

It is easily shown by geometry that this point, the intersection of the three lines drawn from the vertices of a triangle to the middle points of the opposite sides, may be found as follows: Draw from any angle a line bisecting the opposite side, and take a point in that line two-thirds of its length from the angular point. The point thus found is the centre of inertia of a uniform triangular lamina or plate.

(4.) *A triangular pyramid.*—The centre of inertia is in the line drawn from the vertex to the centre of inertia of the base, and at the point in this line distant from the vertex three-quarters of the whole length of the line.

(5.) *Uniform sphere or spherical shell.*—At the centre. The same applies to a series of concentric spherical shells, each of which is uniform throughout, but which differ one from the other in density.

(6.) *Uniform cube or parallelopiped.*—At the intersection of the diagonals.

(7.) *Uniform cylinder.*—At the middle point of its axis.

(8.) *Right Cone* of uniform material.—In the axis of the cone, a quarter of the height of the vertex from the base.

CHAPTER V.

EQUILIBRIUM OF SIMPLE MECHANICAL ARRANGEMENTS.

40. There are six elementary machines which are frequently called the six mechanical powers. Some of these form excellent examples of the application of the principles of statics. The six mechanical powers are—

the Lever, the Wheel and Axle, the Pulley, the Inclined Plane, the Wedge, and the Screw. Of these the wheel and axle is only a modification of the lever, and the wedge and screw are modifications of the simple inclined plane.

41. Equilibrium of a Body turning round a fixed Axis—Levers.—A lever is a rigid bar turning round a fixed point or axis; the fixed point round which the lever turns is called the *fulcrum*. When there are only two forces applied to the lever, one of them is generally called the power, and the other is called the resistance or weight; and levers are commonly distinguished into three kinds,* according to the relative positions of the power, the resistance, and the fulcrum.

In one kind, the power and the resistance are one of them at each end of the lever, and the fulcrum in the middle, as in the diagram, fig. 41. The common balance is an instance of this kind (though the names power and resistance are not applicable).

In another kind of lever (fig. 42), the resistance is applied at a point between the fulcrum and the power.

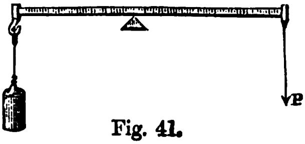

Fig. 41.

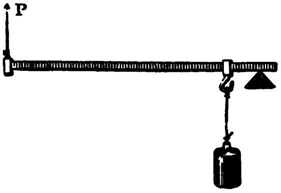

Fig. 42.

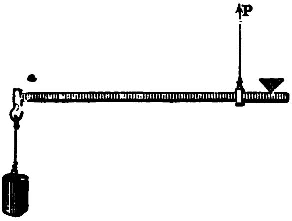

Fig. 43.

In the third kind of lever (fig. 43), the power is applied between the fulcrum and the resistance. A most important example of this kind is to be found in the human fore-arm.

---

* Sometimes, but uselessly, called the first, second, and third *orders* of levers, the arrangement being that in the text.

The equilibrium of a lever, under any forces, may be considered in two ways. One way of dealing with the problem is to find the resultant of all the forces applied to the lever, and to determine whether the line of action of the resultant passes through the fixed point or fulcrum. If it does so, there will evidently be equilibrium, assuming, of course, that the fixed point is able to bear the pressure upon it. Generally, however, it is more convenient to apply the principle of moments explained in Art. 30.

Considering here a lever turning round an *axis*, and thus confined to move in one plane, which is the case of most importance, the principle may be stated as follows: The body will be in equilibrium if the sum of the moments, about the fulcrum, of the forces tending to turn the body in one direction, is equal to the sum of the moments, about the fulcrum, of the forces tending to turn the body in the opposite direction.

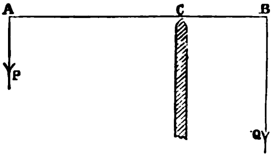

Fig. 44.

Consider, in the first place, a weightless lever with two parallel forces applied at its ends. Let C be the fulcrum on which AB turns. Then if

$$P \times AC = Q \times BC,$$

or if the moments of the forces about C are equal and are opposite in direction, the resultant of the forces passes through C, and the lever is in equilibrium. If the forces are not parallel (fig. 45), then again, as we have seen (Art. 30), if, when we draw C*m* and C*n* perpendiculars on the lines of the forces,

$$P \times Cm = Q \times Cn,$$

that is, if the moments of the forces about C are equal and in opposite directions, the resultant of the two forces passes through C, and the lever is in equilibrium. When the weight of the lever itself is to be taken into account, it may be considered to act through the centre of gravity of the lever, and vertically downward. Its moment about the fulcrum is then added in, according to the direction in which it tends to turn the body. The same must be done with the other forces, if there be more than two, applied to the body. When it is required to find the pressure on the fulcrum, it is necessary to determine the magnitude of the resultant of all the forces applied to the lever, including the weight of the lever itself. This must be done by the methods already given for the composition of forces. The distances of the power and resistance from the fulcrum are often called the *arms* of the lever. From what has been said, it is plain that where the power and resistance are applied in parallel lines, *the power and resistance are inversely proportional to the lengths of the arms of the lever.*

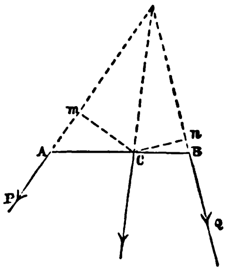

Fig. 45.

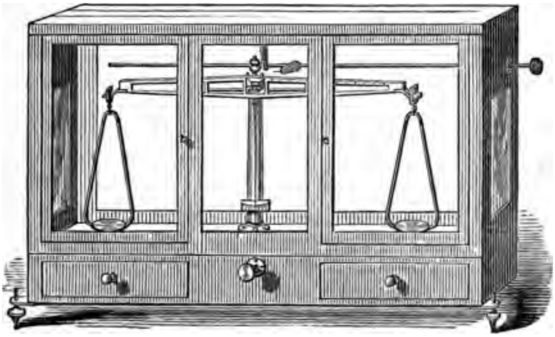

Fig. 46.

42. The Balance.—We may now consider the balance as an important example in statics. Fig. 46 represents a chemical balance. Great care has been bestowed by philosophical instrument makers in making the chemical balance as perfect as possible. The beam is an open framework of brass, made in such a way as to give the greatest possible lightness compatible with the necessary stiffness. The beam turns on a *knife-edge* of hardened steel or agate, resting on a polished agate plate, which is supported on a strong vertical pillar of brass. At each end of the beam there is a steel knife-edge turned upward; and the pans are supported by being attached to agate plates which rest on these knife-edges. In order to prevent injury, and to save the knife-edges from wear, there is a framework which can be raised up, so as to lift the beam off its knife-edge, and to support the weight of the scale-pans, both when the balance is out of use, and also while the pans are being loaded or unloaded. This framework is raised or lowered by means of an eccentric connected with the milled head, shown in front of the balance case.

A balance must have its index in the middle position when the weights in the two scale-pans are equal (the scale-pans themselves being equal in weight); it must also be *sensitive;* and, finally, it must have sufficient *stability*, or quickness of return when slightly disturbed, to enable weighings to be performed with the necessary rapidity. We shall consider the first two conditions: to investigate the third it is necessary to determine the time of a small oscillation, a problem which cannot be taken up here. In order that the balance should be poised when it has equal weights at the ends of the beam, it is necessary that the lengths of the arms should be equal, and that the centre of gravity of the beam should be in the same vertical plane with the axis of support when the beam is horizontal. In figure 47 is shown the arrangement of the points of application of the forces applied to the balance when the pans are loaded. O is the point of support, or knife-edge, on which the beam turns. G is the centre of gravity of the beam, and A is the point where the line OG cuts the line BC that joins the points of support of the scale-pans. OGA is perpendicular to BC. Thus, G and A are in the same vertical through O when the balance is poised and the index OI in the middle position; and these points are shown as being *below* the point O, which ensures the necessary stability of equilibrium. See Art. 49. This being the case, and the lengths of the arms being equal, it is plain that equal weights at the ends will counterbalance each other, and the beam will rest horizontal. The moments of the forces about the fulcrum are then equal and opposite.

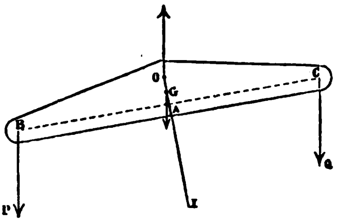

Fig. 47.

The best way of ascertaining whether a balance is true or not is to weigh the same article first in one scale pan and then in the other. Any inequality will be thus detected. When a balance has unequal arms there are two ways of obtaining true weighings with it. The first way is simply to counterbalance the article to be weighed with shot or otherwise; then, removing the article from the pan, to put in weights sufficient to counterbalance the shot. The second method is to weigh the article first in one pan of the balance, and then in the other. If we multiply together the numbers that express the two apparent weights and take the square root of the product, we obtain the true weight of the article. To prove this, let a and b be the lengths of the arms, and let w be the true weight of the article; let x be the apparent weight when the article is suspended at the end of the arm a, and y the apparent weight when it is suspended at the end of b. Then according to the principle of moments, when there is equilibrium we have during the first weighing

$$w \times a = x \times b;$$

during the second,

$$w \times b = y \times a;$$

and, multiplying together the left hand members, and also the right hand members, of these equations, and removing the common factor $a \times b$, we get

$$w^2 = x \times y.$$

Hence $w = \sqrt{xy}$.

The sensibility of the balance is measured by the angle through which a given *percentage* difference of loads in pans will turn the beam of the balance. If the weight P be greater than the weight Q, the beam of the balance will not rest horizontal, but in such a position as that shown, fig. 48. Observing that it is the upward pressure of the support at O which counterbalances the downward forces through B, C, and G, it is plain that O is a point on the resultant of the latter three forces; and thus, according to Art. 28, the moments of those forces about O must balance. Thus the moment of P about O balances the sum of the moments of Q, and of the weight of the scale-pan acting downwards through G, about the same point. In fig. 48, BC is the straight line joining the points of support of the scale-pans. O is the fulcrum, and since G, the centre of gravity of the beam, must be vertically under O when BC is horizontal, the line OGA must be perpendicular to BC. It is found by application of dynamical principles to the problem that the sensibility is greater *the greater the length of the beam BC, greater as the weight of the beam itself is less, greater the shorter the distance OA, that is, the less the distance from the fulcrum to the straight line joining the points of suspension of the pans, and greater the shorter the distance OG, that is, the shorter the distance of the fulcrum from the centre of gravity of the beam.*

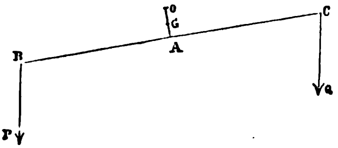

Fig. 48.

43. The Steelyard consists of a beam of iron, balancing about a fulcrum or pivot F. One of the arms is short, and carries a scale-pan; the other, which is longer, is graduated, and has a movable weight sliding along it. It is so adjusted that when the weight is at zero, on the graduated arm, the beam equilibrates about the fulcrum with the scale-pan empty. In order to equilibrate a weight put into the scale-pan, the movable weight is put at a proper distance from the fulcrum, and by means of marks on the beam the weight in the pan is read off. It is plain that the distances between these marks must be equal if they are to correspond to successive equal additions of weight in the scale-pan; for if the movable weight placed at one inch from the zero counterbalances one pound in the scale-pan, the same weight at two inches from the zero will counterbalance two pounds in the scale-pan. In fact, by considering the moments of the forces round the fulcrum we see that a constant weight at double, triple, quadruple, etc., distances from the zero counterbalances a double, triple, or quadruple weight placed on the other arm of the beam, always at a constant distance from the fulcrum.

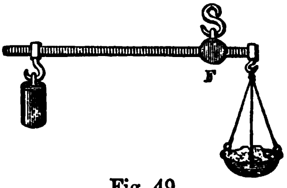

Fig. 49.

The complete working out of this question is left as an exercise to the student. He must take into account the weight of the scale-pan, and the zero position of the movable weight on the arm. The weight of the beam, and the position of its centre of gravity must also be taken into account unless, for simplicity, the latter is assumed to be at the fulcrum.

44. The Wheel and Axle.—This arrangement (fig. 50) consists of two forces applied, one of them to a small cylinder called the axle, the other to a large cylinder called the wheel, the two cylinders turning round the same axis. Taking moments about this axis, it is evident that when there is equilibrium the forces are

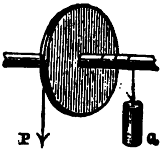

Fig. 50.

inversely as the radii of the wheel and axle, respectively, and that this holds whether the forces are acting in parallel lines or not. The windlass, sometimes used for drawing water from a well, is an arrangement equivalent to the wheel and axle. So is the capstan, commonly employed on ship board. In the case where the rope is wound up on a drum, account must, of course, be taken of the ever increasing radius of the axle.

45. The Inclined Plane.—A body rests in equilibrium on a plane of given inclination under the action of a force which acts parallel to the inclination of the plane. It is required to find the force necessary to prevent the body from slipping down, and also the pressure on the inclined plane, there being no friction.

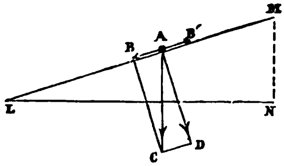

Fig. 51.

Let A represent the body, acted on by gravity, vertically downward in the direction AC, and kept from slipping down the inclined plane by a force in the direction AB′. Let us cut off AC to represent the magnitude of the force of gravity on A. Thus, if A is five pounds, AC may be made $\frac{5}{10}$ of an inch on a convenient scale. Resolve the force AC along the plane, and perpendicular to it (Arts. 26 and 14). Thus, from A draw AB and AD along the plane, and perpendicular to it; and from C draw CB and CD, perpendicular to those two lines, and completing the parallelogram ABCD. AB and AD are the components required. Now AD is perpendicular to the plane, and is the pressure upon the plane due to the gravity of A. AB acting parallel to the plane exerts no pressure on the plane, but simply tends to draw A down the plane, and it is this force which must be equilibrated by the force AB′. The respective magnitudes of AD, the pressure on the plane, and of AB′, which is equal to AB, may be found as in Art. 26, if the *inclination* of the plane is given. They may also be found by calculation.[*] From any point M draw the vertical MN. The lengths LM, MN, and LN, are respectively called the length of the plane, the height of the plane, and the base of the plane. Then because the triangles CAB and LMN are similar,

AB : AC :: MN : ML.

Thus the *force along the plane is to the gravity of A as the height of the plane is to the length.* Again, since

AD(=CB) : AC :: LN : LM,

we see that *the pressure on the plane is to the gravity of A as the base of the plane is to the length.*

If the force which supports the body on the inclined plane acts obliquely to the plane, instead of parallel to it, it must be resolved into two components, one of them along the plane, and the other perpendicular to it. The former of these components prevents the body from slipping down the plane, the latter has its effect in diminishing the pressure on the plane, or increasing it, as the case may be. The accompanying diagram illustrates this for one case. The dotted lines are the components of the force F, communicated to A by a string passing over a pulley. The full lines are the components of the gravity of the body A.

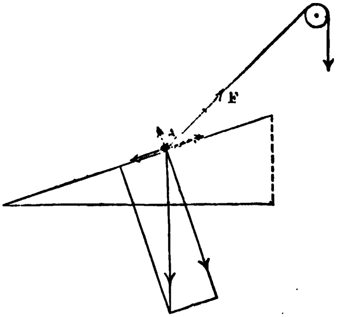

Fig. 52.

46. The Wedge and Screw are, as has been already

---

[*] Trigonometrical solution. Let i be the inclination of the plane to the horizontal. Let the gravity of A be W pounds, let F be the force AB or AB′, and let P be the pressure on the plane. Then

$$F = W \sin i, \text{ and } P = W \cos i.$$

remarked, modifications of the inclined plane. Figure 53 shows a form of wedge, which is sometimes called the movable inclined plane. The student may see the relation between the screw and the inclined plane thus: Let him take a triangular piece of paper, right-angled at C, as shown in fig. 54, and, placing the side BC against the edge of a pencil, or other uniform cylinder, let him roll it on the pencil, when the edge AB of the paper will form a screw on the cylinder. We shall not here attempt the investigation of the equilibrium of the frictionless wedge or screw. The problem would not present the simplicity afforded by the problem of the frictionless inclined plane; and the problem of the wedge or screw in action, taking friction into account, is one of which only a very rough, incomplete solution can be given.

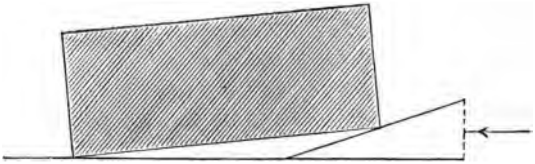

Fig. 53.

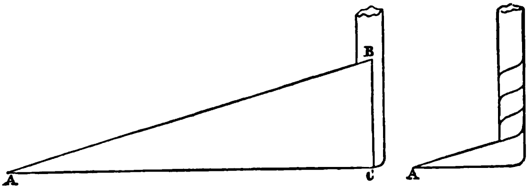

Fig. 54.

47. The Pulley.—Theoretically, the pulley is a smooth body which changes the direction of a flexible and inextensible string that passes over it and presses upon it. Practically, to avoid friction as far as possible, the pulley is a wheel, and the cord passes over the circumference of the wheel. If the cord were perfectly flexible, and if there were no friction between it and the pulley, then the tension of the cord would be the same throughout. Hence, in fig. 55 (1) two equal weights attached at two ends of a cord passing over a pulley balance each other. So also, in fig. 55 (2), the force that must be applied to the unattached end of the cord, in order to support the ball hung at the other extremity, is equal to the gravity of the ball. In this diagram the pulley is simply used for changing the direction of the force applied to support the ball. In figs. 55 (3) and (4) a pulley is employed for the purpose of diminishing the force required to support the weight. Let P be the pull applied to the unattached end of the cord. Then, as has been said, if there is no friction, the pull in every part of the cord is the same; and therefore in the part attached to the hook of the beam. Thus the ball is supported by two parallel forces each equal to P;* and the force P required to be applied to the cord, in order to support the ball, is only half the

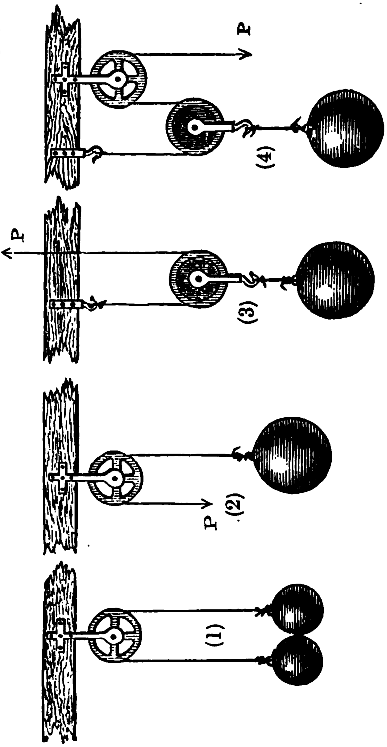

Fig. 55.

---

* Here the weight of the pulley and cord are left out of account.

gravity of the ball. In fig. 56 (5), where three movable pulleys are employed, it will be seen that the pull in the cord that passes round b_1 is equal to one-half of the gravity of the weight supported. The pull in the cord passing round b_2 is one-half of that in the first cord, or one-fourth of that required to support the ball; and the pull in the cord passing round b_3, which is equal to the force P, is similarly one-eighth of that required to support the ball.

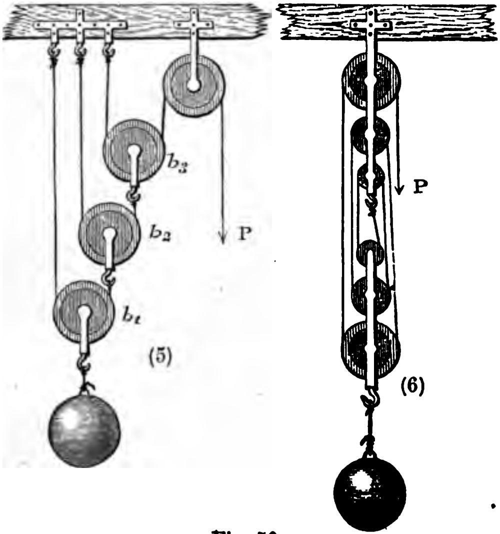

Fig. 56.

If there are n movable pulleys, arranged in the same manner, all the cords being parallel, then the force P, required to be applied to support a weight W, is given by the equation:

$$P = \frac{W}{2^n}$$

When a complex pulley, such as that shown in fig. 56 *(6), is used,* a single cord passing round all the wheels, and all the parts parallel, then considering that the upward pull in each cord is the same and equal to P, and counting up the number of strings affording support to the weight, the relation between P and the weight may be found. Thus in fig. 56 (6),

$$P=\frac{W}{6},$$

the weight of the pulley and the cord, and also any slight deviation from parallelism of the parts of the cord being here left out of account.

48. Equilibrium of a Body Resting on a Horizontal Plane.—A rigid body resting on a horizontal plane touches the plane in three or more points. Theoretically, a rigid body may rest on a rigid plane on only three points: but we have no perfectly rigid bodies; and, each of the two bodies yielding to some extent, there are always, in practice, considerable surfaces in contact. The gravity of the body resting on the plane is equilibrated by the upward pressures through the points of support. The resultant of all the upward pressures must be equal to the gravity of the body. It must also act in the line in which the resultant of the gravities of the several particles of which the body is composed acts; that is, it must act upward in the vertical through the centre of gravity of the body. From this it will be seen that one condition for the equilibrium of a body set down on a horizontal plane is that the downward vertical through the centre of gravity should fall *within* the polygon, formed by drawing lines joining the most outlying points of support of the body. For the resultant of all the upward pressures must obviously act through a point within the polygon so formed; and thus the line of action of the gravity of the body, which is the equal and opposite of this

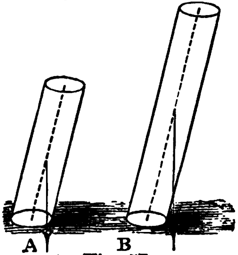

Fig. 57.

resultant, must also fall within that polygon. Thus in the diagram an oblique cylinder A rests in equilibrium on the horizontal plane, because the line through its centre of gravity falls within the ellipse drawn round the base; but the cylinder B, which is taller, will not stand in equilibrium, because the vertical through the centre of gravity falls without that ellipse, and cannot be equilibrated by the resultant of upward pressures on the base of the cylinder.

49. Stable, Unstable, and Neutral Equilibrium.— When a body, under the influence of gravity,* is supported in equilibrium, the equilibrium may be stable, unstable, or neutral. The equilibrium is said to be *stable* if, when the body is very slightly (properly infinitely little) displaced from the position of equilibrium, it tends to return to that position. It will then, if left to itself, vibrate about its position of equilibrium, and, finally, resume that position when the vibration has subsided. The equilibrium is *unstable* if, when the body is very slightly displaced from its position of equilibrium, it tends to fall away farther and farther from that position of equilibrium. The equilibrium is *neutral* when the body neither tends to fall away nor to return. Thus a rod, hung up by one end from a hook, is in stable equilibrium; being displaced it tends to return to its original position, and let go, it vibrates like a pendulum. A pointed rod, balanced on one end, which rests on the ground is in unstable equilibrium; being displaced its tendency is to fall farther and farther away from its original position.

A uniform cylinder and a loaded cylinder afford excellent examples of the three cases. B is a uniform cylinder of wood, A and C are wooden cylinders, each loaded with a cylindrical bar of lead which is placed parallel to the axis of the wooden cylinder, and at a distance from the centre. A is in stable equilibrium, resting on a horizontal plane with the loaded part lowest. C is in unstable equilibrium, *balanced* with its loaded part highest. B, on the other hand, rests in neutral equilibrium on a horizontal plane; if it be slightly disturbed from its original position, it has no tendency either to return to it or to fall farther away.

---

* The student will not fail to observe that this is but a particular case of the more general one in which the body is under *the influence* of any forces whatever.

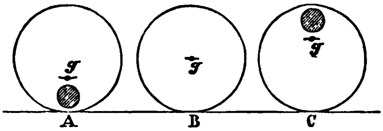

Fig. 58.

Now, it will be noticed that *g* being the centre of gravity in each of the three cases, in the case of stable equilibrium the path of *g*, when the body is disturbed from equilibrium, is such as to make that point *rise* against gravity; and when the body, left to itself, is returning towards the position of equilibrium, the centre of gravity falls till it assumes the lowest position. In the case of neutral equilibrium, the path of the centre of gravity, viz., the centre of the cylinder, is a horizontal line, so that the centre of gravity neither rises nor falls. In the case of unstable equilibrium, the path of the centre of gravity during slight disturbance is such as to lower that point, and the centre of gravity tends to fall until it reaches the lowest position, that in figure A.

The relations that have just been illustrated form the criterion of stability of equilibrium. Noticing that a body may have such a form as to have several positions of stable and unstable equilibrium, as in the case of a polygonal prism, and in many others, it will be seen that a position of stable equilibrium is one of minimum elevation of the centre of gravity, and a position of unstable equilibrium is one of maximum elevation of the centre of gravity, the displacements considered being very small displacements.

It is to be observed that a body may be stable for angular displacements in one direction, and neutral or unstable for displacements in another. Thus an egg-shaped body, lying on a horizontal plane, is stable for angular displacements round the centre of its principal vertical elliptic section through the centre, and neutral for displacements round the centre of its vertical circular section through the centre. Other instances will readily suggest themselves to the student.

Another test of stability, found in the consideration of work done in displacing the body must be mentioned, but cannot be discussed in detail here. It will be noticed that in the displacement of the body A from the stable position of equilibrium, work must be performed to raise the centre of gravity g. In the case of unstable equilibrium, the body itself does work as the centre of gravity falls. In the case of neutral equilibrium, energy is neither gained nor lost during the displacement.

CHAPTER VI.

ON FALLING BODIES.

50. Galileo's Experiments.—Bodies falling through the air often fall with unequal velocities. If a sovereign and a feather, for example, be let fall at the same instant from the same point, the sovereign reaches the ground much sooner than the feather. Galileo was the first to prove that it is the resistance of the air that causes the unequal rapidity of falling, though the explanation was conjectured long before, as early even as Lucretius. Galileo showed that the feather is more retarded than the piece of gold because it offers a greater surface to the air in proportion to its weight. He made experiments to prove that the force of gravity acts equally on equal masses of whatever material, and inferred that all bodies would fall at the same rate were they unresisted by the air. His experiments were made by dropping balls of various materials simultaneously from the leaning tower of Pisa. He found that they all reached the ground almost at the same instant. When, however, the forms of the bodies were altered so as to increase the surface exposed to the air, the times of falling became very unequal.

Since the invention of the air-pump, direct experiments on this subject have become easy. In the celebrated "guinea and feather" experiment, a tall tube is placed vertically on the plate of an air-pump, and is carefully exhausted of air. A sovereign and a feather, previously placed on a drop-stage at the top of the tube, are liberated at the same instant by means of a mechanism working through an air-tight stuffing box. Falling through the vacuum they are seen to reach the bottom at precisely the same instant. Such experiments as these show the effect of the resistance of the air on falling bodies.

Having found that the force of gravity acts equally on all bodies, or, in other words, that equal masses or quantities of matter, of whatever material, are equally attracted by the earth, Galileo next proceeded to investigate experimentally the numerical laws that govern the falling of bodies near to the earth's surface. The velocity of a body falling freely is so great as to make it difficult of observation, and it is necessary therefore to use some contrivance for diminishing the rate of motion without altering the character of the motion, unless, as in Morin's apparatus described, Art. 52, some method of recording the motion automatically is made use of. Galileo used the inclined plane; the apparatus now more generally employed for examining these laws experimentally is Attwood's machine.

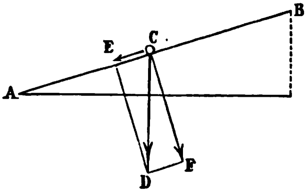

Fig. 59.

Let AB be an inclined plane of such gentle inclination that the motion of a body rolling downwards is easily observable. Let C be a ball rolling down it. The force of gravity on the body C acts vertically downwards in the line CD; but resolving the force CD along the plane and perpendicular to it, we see that it is only the part CE, parallel to the plane, that is effective in producing the motion. The part CF, perpendicular to the plane, is equilibrated, and has its effect merely in pressing the body against the plane. From the figure we see that the force CE, moving C along the plane, is to CD, the whole weight of C, in the same proportion as the height of the plane to the length; and thus, by making the height small enough in proportion to the length, that is, by sufficiently diminishing the inclination of the plane, we can make the force that moves C as small as we please, and therefore the motion as slow as we please. The laws of the motion will not, however, be altered. It is merely in amount, not in character, that the motion is changed.

By observing in this way, Galileo showed, (1) that *a falling body acquires a gradually increasing velocity, the velocity acquired being directly proportional to the time;* (2) that *the spaces described in different intervals of time by a falling body starting from rest are in proportion to the squares of the times;* (3) that *the space described in any time by a body falling from rest is equal to half the space that would be described in the same time with the velocity that the body has at the end of the time of falling.*

51. Attwood's Machine.—The laws of falling bodies may also be examined by means of Attwood's machine, which, like Galileo's inclined plane, is an arrangement for diminishing the amount of the motion without altering the laws of it. The principle of Attwood's machine is to cause the gravity of a small body falling to act upon a large mass free to move. As we have seen (Art. 18), the velocity acquired under the action of a given force for a given time is inversely proportional to the mass moved. Thus when the falling body is connected to a considerable mass that requires to be set in motion, the rapidity of the fall is diminished sufficiently to allow it to be observed conveniently. The laws of the motion remain unchanged; but the velocity at every point, and the space described during each interval of time, are diminished to such an extent as to make them easily observable.

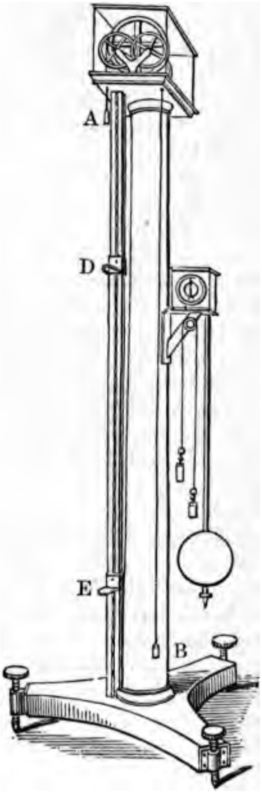

Fig. 60.

The arrangement is the following: — There are two metal boxes, A and B (fig. 60), which can be loaded to any required extent with masses that are put into them. These are connected by a fine silk cord that passes over a wheel supported on *friction-rollers*, which almost do away with resistance of friction. The weights of A and B are equal, and equilibrate each other. Another small weight, made in the shape of a thin flat bar, can be laid horizontally on the top of the cylindrical box A. This is the falling body. (For brevity and distinction we shall call it the "bar.") Now as the bar falls, its gravity, which is the moving force, has to set in motion not only its own mass, but also the masses A and B, as well as the wheel-work; and, by sufficiently increasing A and B, the motion is reduced to any required extent. There are two stages, D and E, shown in the diagram. One of these, D, is a horizontal ring through which the cylindrical box A passes; and, as it passes, the bar is lifted off the top of the box, and its gravity ceases to act on the system. E is a flat plate on which the box A strikes, and downward motion is thus stopped. D and

E are movable, the bar can thus be lifted off at any required point of the course by means of D, and, by adjusting the position of E, the subsequent course can be made of any required length. There is a pendulum, beating seconds or half seconds, attached to the apparatus for the purpose of counting time during the experiment.

The vertical rod of wood which supports D and E is a scale graduated to inches or centimetres. By means of it the positions of the stages are adjusted.

To experiment with Attwood's machine, the box A is drawn up to the top of the scale, the top of the box being parallel with the mark zero. The small horizontal bar is laid on the top of A; and then at a particular beat of the pendulum the system is released. To avoid giving any impulse or shake in releasing the system, there is often a mechanical detent connected with the apparatus by the letting go of which the experiment is commenced. Otherwise the part of string that holds up B may be held with the fingers, or other means may be adopted. The stages D and E, before the experiment commences, are placed at any chosen points. The beats of the pendulum are observed by the ear, to which a little practice gives considerable accuracy, and the time of reaching D is noted. The distance from the zero to D being known, the space fallen through in the observed time is ascertained. At D, as has been explained, the horizontal bar is lifted off, and the distance from D to E is then traversed with unaccelerated motion of the system. By knowing the length DE, and observing by the clock the time taken to traverse it, the velocity that the system acquired while the horizontal bar was falling from the top to D is ascertained.

By means of Attwood's machine, the laws of Galileo given above may be verified experimentally in an admirable way; and with it we may also determine acceleration due to gravity, or the intensity of gravity per unit quantity of matter,* by an elementary experiment. To determine the numerical value of the intensity of gravity by Galileo's method of allowing a body to roll down an inclined plane involves considerations somewhat complicated. In fact, a sphere and a cylinder will not roll down an inclined plane at the same rate, nor will either of them roll down at the rate at which a body would slip frictionlessly down a plane of the same inclination. The calculation of the numerical value of the intensity of the force of gravity by means of an experiment with Attwood's machine is, however, simple, and with well-made apparatus considerable accuracy is attainable.

---

* See foot-note, p. 121.

To explain this—we have seen that when a body, such as the bar of Attwood's machine, falls *freely*, the weight or gravity of the body requires to set the mass of the body itself into motion. But in the experiment described the same force, namely, the gravity of the falling bar, is required to set not only its own mass in motion, but also the masses of the two boxes, together with the mass of the wheel-work. The velocity acquired in any given time by the moving system in Attwood's machine will therefore be (see Art. 18) to the velocity that would be acquired by the bar if it were falling freely, inversely as the whole mass moved in Attwood's machine is to the mass of the bar.

Let *p* be the mass of the bar; let M be the mass of each of the boxes A and B; and let I be a mass equivalent in *inertia for rectilinear motion* to the inertia of the wheel-work,* the different parts of which, it will be observed, are put in motion with very different velocities. Let *g* denote the velocity acquired in one second by a body falling freely, that is, Art. 15, since the present is a case of uniformly accelerated motion, the *acceleration* due to gravity, and let a be the acceleration of the whole mass moved by the weight of p in the Attwood's machine. Then according to what has been said, g is to a *inversely* as the mass p is to the mass

---

* We cannot explain here how the equivalent I is obtained; but it is sufficient for the present that a mass I can be determined such that if we could make the wheel-work perfectly massless (*i.e.*, without *inertia*), and add one-half of I to each of the boxes A and B, we should have a system precisely equivalent to the actual system so far as regards the motion that we are here concerned with.

$$p+2M+I$$

Or,

$$g : a :: p+2M+I : p,$$

Hence

$$g=\frac{p+2M+I}{p}a.$$

The acceleration a, or the velocity acquired by the system under the gravity of p in one second, is observed in the experiment, and g is calculated by this formula.*

52. Morin's machine for observing the laws of falling bodies also requires notice. In it there is a vertical cylinder or drum covered with white paper which is turned at a very rapid and uniform rate. The falling body is allowed to drop down between two vertical guides, and it carries a pencil which touches the paper drum, and thus records the position of the falling body at every instant. On unrolling the paper from the drum, the trace thus made is found to be a parabola. This, it can easily be shown, ought to be the case in order that the laws of Galileo may hold; and by comparing the dimensions of the parabola, the rate of motion of the drum being known, the velocity and acceleration of the motion at every point of the course of the falling body may be inferred. The figure shows the paper after it has been unrolled from the drum. P is the point from which the body is let fall, let us suppose, just at the moment when the number 0, carried by

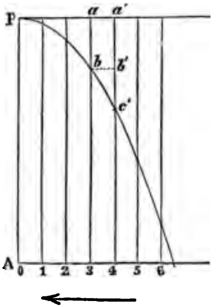

Fig. 61.

---

* Note that if the scale is divided into inches, the value of g given by readings on the scale will also be in inches.

the paper, turning rapidly round in the direction indicated by the arrow, comes to be vertically under P. Were the paper at rest, the trace made by the body falling would be the line PA; but as the paper has a motion of its own, the line PA is drawn forward during the fall, and the line traced is found to be a curve of the form shown, a portion of a parabola. The equal spaces $0-1$, $1-2$, etc., represent *equal* intervals of time owing to the uniformity of the motion of the paper; for example, each of them may be $\frac{2}{10}$ of a second. When therefore the body has been falling $\frac{6}{10}$ of a second, it is plain that it has fallen through the distance ab. The average velocity during the next $\frac{2}{10}$ of a second is ascertained by measuring the distance $b'c'$ If $b'c'$ be measured in feet, and if the number found be divided by $\frac{2}{10}$ (*i.e.*, multiplied by 5), the average velocity in feet per second will be the result (see Art. 9). Again, by measuring ab and $a'c'$ and comparing them, Galileo's discovery that the spaces described are proportional to the squares of the times during which they are described is verified.

53. We next proceed to deduce convenient formulas for solving the many interesting and useful problems that are connected with falling bodies. A body that falls from rest is, as we have seen, influenced by a force that is sensibly constant, under which it acquires velocity at a uniform rate. A body falling for one second in our latitude acquires a velocity of 32·2 feet per second; during the next second an additional equal increment of velocity is acquired, and so on. Let g denote the number that expresses the acceleration due to gravity, as is usual in books on dynamics;* and let t be the time in seconds during which the body is falling; then v being the velocity acquired:—

$$v=gt. \qquad (1a)$$

If the body has initially a velocity V, that is, if it be projected vertically downward or upward with a velocity V, the velocity at the end of any time t is calculated by adding or subtracting, as the case may be, the velocity acquired. Thus we have the formulas—

---

* Thus g is 32·2 feet per second gained per second, or 981 centimetres per second gained per second. (See foot-note p. 121).

$$v = V \pm gt. \qquad (1b)$$

It is to be noticed that in the formula $v = V + gt$ we are taking the downward direction as the positive direction, while the opposite is the case in the formula $v = V - gt$. It is convenient to make the given initial velocity *positive;* and then to add or subtract, as the case may be, the velocity subsequently acquired under gravity.

In the formula $v = V - gt$, when t has a certain value (depending on the magnitude of V), the value of v is zero; when t has any greater value, v is negative. These are, of course, the first, the case where the body rises to the highest point of its course and has zero velocity just before beginning to descend again; and the second, the case where it has begun to fall, and has downward or negative velocity. By putting $v = 0$, and calculating t, we find the time during which a body, projected vertically upward with given velocity, will continue to rise.

Let us next find the space described in any given time by a body falling freely. If the body were moving with uniform velocity, the space fallen through would be found by multiplying the number that expresses the velocity by the number of seconds during which the motion lasts. The formula for calculation would be (see Art. 8) $S = VT$ But when the velocity is not constant we must use instead of V the number that expresses the *average* velocity of the motion during the time. In the case of uniformly accelerated motion, the average velocity is obviously the arithmetic mean between the velocity at the beginning and the velocity at the end of the time. Let the initial velocity of the motion be V, that is, at the commencement let the body be projected downward with the velocity V, and at the end of the time t let the velocity be $V+v$ (where, as we have just seen, $v=gt$). The average velocity of the motion is then

$$\tfrac{1}{2}\Big[V+(V+v)\Big] \text{ or } V+\tfrac{1}{2}v.$$

Consequently, if s be the space described, it is found on multiplying the average velocity by the time t that

$$s=(V+\tfrac{1}{2}v)t;$$

and substituting gt for v,

$$s=Vt+\tfrac{1}{2}gt^2. \qquad (2b)$$

If the body falls from rest the velocity of projection, V, is zero. The formula then becomes

$$s=\tfrac{1}{2}gt^2, \qquad (2a)$$

which corresponds with Galileo's discovery that the spaces described by bodies falling from rest are in simple proportion to the squares of the times of falling.

If the body is projected vertically upward, the average velocity of the motion is

$$V-\tfrac{1}{2}v \text{ or } V-\tfrac{1}{2}gt,$$

and the space described is given by

$$s=Vt-\tfrac{1}{2}gt^2. \qquad (2b)$$

From the last formula we may calculate the greatest height to which a body will rise with a given velocity of projection, having previously determined, with the aid of (1*b*), the time during which the body rises. We may also calculate the time at which the body returns to the point of projection, in which case,*

$$s=0.$$

---

* To find the value of t from the equation, $Vt-\tfrac{1}{2}gt^2=0$, a quadratic equation must be solved. The two roots, one $t=0$, and the other $t=2\frac{V}{g}$, correspond to the two instants at which the body is at the point of projection, the first at the start, the second on its return. Comparing the latter with $t=\frac{V}{g}$, got by putting $v=0$ in (1*b*), at which time the body reaches its highest point, we find that the time required by the body to return to the point from which it started is double that required to rise to its greatest height. Thus the times of ascent and descent are equal.

Another formula of great importance is derived from the two formulas

$$v = V \pm gt$$
$$\text{and } s = Vt \pm \tfrac{1}{2}gt^2$$

by eliminating t. This may done by squaring the upper, multiplying the lower by $2g$, and subtracting one from the other. The result obtained after transposing $2gs$, is

$$v^2 = V^2 \pm 2gs, \qquad (3_b)$$

or if V, the velocity of projection, is zero

$$v^2 = 2gs, \qquad (3_a)$$

v being measured downward. Thus it appears that the square of the velocity acquired in falling from rest through any distance s is equal to twice the product of the acceleration g and the space s. The formula (3_b)

$$v^2 = V^2 - 2gs$$

enables us to determine the height to which a body, projected vertically upward with a given velocity V, will rise. For putting $v = 0$, which will be the case at the highest point; or

$$V^2 - 2gs = 0,$$

we get

$$s = \frac{V^2}{2g}.$$

Example.—A body projected vertically upward with a velocity of 80 feet per second. We find (putting g roughly equal to 32),

$$s = \frac{V^2}{2g} = \frac{(80)^2}{2 \times 32} = \frac{6400}{64} = 100.$$

The body rises to a height of 100 feet.

54. In formula (2_a), let us put for g its approximate value 32. We find for the number of feet described in t seconds by a body falling from rest,

$$s = 16 \times t^2$$
$$\text{or } s = (4t)^2.$$

Thus, since $4t$ is the number of quarter seconds that the body has been falling, we obtain a rough but convenient rule that *the number of feet traversed by a body falling from rest, is equal to the square of the number of quarter seconds during which it has been falling.*

Example.—A stone is let fall from the top of a cliff, and 3½ seconds (14 quarter seconds) later it is heard to strike at the bottom: required the height of the cliff. From the rule we find

$$H=(14)^2=196 \text{ feet.}$$

In this calculation, besides taking the numerical value of g to be 32, which is less than its real value, we are leaving out of account the influence of friction of the air on the falling body, and the small correction to be applied for the time taken by the sound to reach the top of the cliff.

In the formula (2_a) let us take $t=1$, $t=2$, etc. Thus we get

	sec.	ft.
Space described from rest in	1	16
,, ,,	2	16 × 4
,, ,,	3	16 × 9
,, ,,	4	16 × 16

Subtracting the number of feet described in one second from the number described in two seconds, we find the space described during the 2nd second, and so for 3rd and the 4th seconds. Thus we find

	sec.	ft.
Space described in	1st	16
,, ,,	2nd	16 × 3
,, ,,	3rd	16 × 5
,, ,,	4th	16 × 7;

and we see that *the spaces described from rest during the successive seconds are in proportion to the successive odd numbers* 1, 3, 5, 7, etc.

55. The principles that have been explained for the case of bodies falling vertically under the influence of gravity, apply to all cases in which a body is in motion under the influence of a constant force acting in the line of motion of the body; and the formulas to be used in all calculations concerning such motion are precisely similar to those that have just been given. Let a body, whose mass is m, be acted on by a given constant force, for a given time t; and let it be required to determine the velocity acquired, and the distance traversed in that time. First, if the force be not numerically expressed in terms of the kinetic unit of force, it is necessary to reduce it to kinetic measurement. This being done, let F be the number that expresses the force in kinetic units; then, dividing F by m, the acceleration a of the moving body is obtained (see Arts. 15, 18, and 22).

Thus

$$a=\frac{F}{m}. \qquad (4)$$

Then, if there be no initial velocity, substituting a for g in formula (1_a), page 101, we obtain

$$v=at \qquad (5)$$

for the velocity at the end of the time t; and, similarly, we have for the space described from rest,

$$s=\tfrac{1}{2}at^2; \qquad (6),$$

and for the velocity acquired while the body is moving over any distance s we have

$$v^2=2as. \qquad (7)$$

If the body be in motion with an initial velocity V, the corresponding formulas (1_b), (2_b), (3_b), are to be employed, a being substituted for g in each case.

56. It often happens that problems of a kind converse to those that we have been considering require to be solved.

Example I.—A train, with its engine attached, weighing 200 tons, is moving with a velocity of one mile per hour, and ten seconds later it is found to be moving with a velocity of 3 miles per hour. It is required to find the accelerating force, supposing it to have been constant during the time of action.* Here the initial velocity is 1·47 ft. per second nearly, and after ten seconds it has been accelerated to a velocity of 4·4 ft. per second. The acceleration in feet per second gained per second was therefore $\frac{2·93}{10}$. The mass moved is equal to 200 × 2240 lbs. = 448,000 lbs. Hence if F denote the accelerating force in British kinetic units, we have from (4)

---

* Friction and the inertia of the wheels are left out of account here. Thus the force calculated is *over and above* that required *to* overcome friction and the inertia of the wheels.

$$F = \frac{2·93}{10} \times 448{,}000 = 131{,}300 \text{ nearly.}$$

Dividing this number by 32·2, according to Art. 22, we find the force to be 4080 lbs., or a force which would support nearly 2 tons.

Example II.—A bullet weighing 1 oz. is fired from the barrel of a rifle $2\frac{1}{2}$ ft. long, and leaves the muzzle with a velocity of 1500 ft. per second: find the *average* pressure on the bullet while in the barrel. The average pressure* is such a pressure as would have communicated the velocity, 1500 ft. per second, to the bullet had it been kept uniformly applied to the bullet over the whole distance—the length of the barrel—along which acceleration was being communicated to the bullet; and thus this problem is reduced to the case of motion under constant force. From formula (7), p. 106, we have

$$\begin{array}{ll} & v^2 = 2as, \\ \text{hence} & (1500)^2 = 2a \times 2\tfrac{1}{2} \\ \text{and} & a = 450{,}000 \end{array}$$

But $m = \frac{1}{16}$ lb., and by (4), p. 106, $F = ma$; hence

$$F = \frac{450{,}000}{16} = 28{,}125 \text{ kinetic units} = 873 \text{ lbs.}$$

---

* There are two ways in which we may consider the average of a force that is not uniform in its action. We may consider a force which, acting uniformly during a certain *time*, would produce a given effect produced in the same *time* by the non-uniform force; or we may, as in the present example, consider a force which, acting over a certain path or *space*, would produce the given effect produced by the non-uniform force acting over the same *space*. The former average is spoken of as the *time-average* of the non-uniform force, the latter as the *space-average* of the non-uniform force.

CHAPTER VII.

CURVILINEAR MOTION.

57. Having considered certain cases of motion in a straight line in the preceding chapter, we have next to consider motion in a path that is curved. According to Newton's first law of motion, a body that is in motion tends to move on from point to point with unvarying velocity in a straight line: and curvilinear motion can only take place under the influence of a force, or forces, always deflecting the body from its rectilinear path. In fact, a body moving in the line AB would continue to move uniformly in that line, did not force act on it to deflect it from that line. This force (represented in direction, at one point, by CE) acts in overcoming inertia, and in communicating to the body the transverse velocity, which, combined with the original velocity, gives the new motion. When the transverse force ceases to act, the body thenceforward moves on in a straight line. It may be noticed here that the deflection of a body from the line AB into a new line, such as CD, cannot be sudden and abrupt. It must be made by a gradual turn. An abrupt turn from the line AB into a new line CD, giving a sharp corner at C, would necessitate the action, for an infinitely short time, of a force infinitely great. The bend must be gradual; and the sharper the bend, that is to say, the shorter the path for a given angle turned through, the greater is the force required to produce the change in direction of the motion of the body.

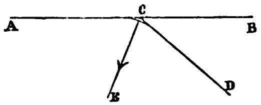

Fig. 62.

As examples of curvilinear motion, we may consider the motion of a stone whirled round in a sling, the motions of the moon and of the planets in their orbits, the motion of a projectile, as a bullet fired from a gun. In each of these cases, a deflecting force is acting on the body at every point of its path. Thus in the case of the sling, the cord is applying to the stone a force directed toward the centre, round which the motion is taking place; and as soon as the stone is released from the sling it flies forward in a straight line, the tangent to its path, at the instant of release. The moon is drawn towards the earth, and the planets towards the sun, by the force of gravitation. If the force of gravitation ceased to act, the moon would cease to move in a curve, and would go on thenceforth in a straight line, the tangent to her path, at the instant at which gravitation ceased acting.

The case of the projectile which, under the influence of gravitation, is always falling towards the earth is similar. It will be briefly considered a little later.

In whirling a stone round in a sling, we feel that the hand is required to apply force in order to make the motion curvilinear. On account of the *inertia* (see Art. 19) of the body, resistance is exerted by the body against the deflecting force; and it is this which gives rise to the outward pull in the cord of the sling that is felt by the hand. The pull felt by the hand, when a stone in a sling is whirled round, is called *centrifugal force;* but this term is by no means appropriate, and is apt to be misleading. There is no real force tending to make the body *fly from* the centre. What we observe is simply the tendency of the body to move on in the direction, whatever it may be, in which at any instant it is moving. It is only when the body is compelled by a material guide, such as the cord of the sling, to move in a curved path that any resistance to the force that gives curvature to the path is experienced. Thus, in the case of a planet, the body *yields* to, and is deflected by, the force transverse to the path; and in this case, though the name centrifugal force is commonly applied, it is evidently unsuitable.

58. Uniform Circular Motion.—The simplest case of curvilinear motion is uniform circular motion. This is the case of a body moving in a circle with unvarying velocity. The following are the laws of uniform circular motion:—

1. To keep a body moving uniformly in a circle, it must be acted on by a force always directed towards the centre of the circle. Considering that the path of the body at each point is in the direction of the tangent to the circle at that point, we see that this *centreward force* is at every point perpendicular to the path of the body.

2. The centreward force is constant, that is, it is the same in amount at every point of the path. This will be almost self-evident from considering the uniform character of the motion about the centre.

3. Let m be the mass of the body moving in a circle of radius r, let v be velocity, that is, the length of the curved path described per unit of time; then F, the amount of the centreward force, reckoned in kinetic units (Art. 21), is given by the formula

$$\mathrm{F}=\frac{mv^2}{r} \qquad (1)$$

Reducing to gravitation units (Art. 22), the force P is found by

$$\mathrm{P}=\frac{mv^2}{g\times r} \qquad (2)$$

Thus *the force required to keep a body of given mass revolving in a circle of given radius, is proportional to the square of the velocity of the motion. For a given velocity the centreward force is inversely proportional to the radius, and is directly proportional to the mass of the body.*

It will, of course, be observed that the so-called centrifugal force is equal and opposite to what is here called the centreward force. *Its magnitude is thus to be calculated by the same formula.*

Example.—A ball 10 pounds weight, let us suppose one of the balls of the governor of a steam-engine, is turning in its circle with a velocity of 3 feet per second, the radius of the circle being half a foot. According to the formula given, the force is equal to

$$10 \times \frac{(3)^2}{\frac{1}{2}} = 180 \text{ British kinetic units, poundals;}$$

or, dividing by 32·2, 5·6 pounds force.* Thus to keep the ball revolving in its circle, it must be constantly urged by a force directed towards the centre of the circle, and equal in magnitude to the gravity of 5·6 pounds.

59. The laws of uniform circular motion may be put in other forms that are often convenient. Let T be the time taken by the moving body to describe one complete round. This is sometimes called the *periodic time* of the motion. The radius of the circle being r, the length of the circumference is equal to $2\pi r$, where π stands for the ratio of the length of the circumference of a circle to the length of the diameter, and is numerically equal to 3·1416.† Now, as T is the number of seconds required to describe the length $2\pi r$, we have v the velocity given by

$$v = \frac{2\pi r}{T}.$$

Substituting this in formula (1) above, we find, if F denote the force in kinetic units,

$$F = m\frac{4\pi^2 r}{T^2}. \qquad (3)$$

Thus *the force required to keep a body moving in a circle with given periodic time is directly proportional to the mass, and directly proportional to the radius of the*

---

* The force will be reckoned in poundals or in dynes, according as the mass and the length of the radius are taken in British or in metrical units; and the reduction to pounds force or grammes force will be performed according to the rule of Art. 22. It will always be proper for the student to perform the reduction; for, while we help ourselves to clearness and exactness by using the absolute system, we must be prepared to give the results, if they are to be of practical use to others, in terms of the units commonly employed by engineers and other practical men.

† This will be found explained in books on trigonometry and mensuration.

circle; while the forces required to keep the same body moving in the same circle with different periodic times are inversely proportional to the squares of the periodic times.

Lastly, let us consider the *angular* velocity of the motion. Let A be a body revolving in a circle ABC. Let A start from C, and commence to revolve uniformly in the circle. The rate at which the radius OA sweeps over the angular space is called the angular velocity of the body A. Angular velocity, when uniform, is measured by the angle described per unit of time. If it is not uniform, considerations similar to those of Arts. 9 and 10, where non-uniform linear velocity is the subject, are applicable. Just as linear velocity is numerically expressed in feet per second, or metres per second, so angular velocity is numerically expressed in units of angle swept over per second. Angular velocity may be expressed in degrees per second, or again in right angles per second, or in revolutions per second, according as the degree, the right angle, or the turn through a complete round is taken as the unit angle. The usual and most convenient unit angle for the measurement of angular velocity is, however, the radian, or the angle which, at the centre of a circle, subtends an arc of the circumference equal in length to the radius of the circle.[*] The radian is equal to 57·3 degrees approximately, or a little less than the angle of an equilateral triangle. One round (360°), or the angular space of four right angles, is therefore 2π radians; that is approximately 2 × 3·1416, or 6·2832 radians. The radian is therefore contained 6·28 times in four right angles. According

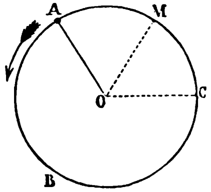

Fig. 63.

---

[*] The name *radian* has recently been given to this unit angle. The angle COM, fig. 63, is one radian. The arc CM is equal to the radius of the circle. Thus, supposing the ball A to be in its first revolution, the eye tells us that its radius AO has described an angle a very little greater than two radians from CO.

to this, if T be the periodic time of the motion, that is, the time of describing a complete round, since 2π is the angular measure of a round, $\frac{2\pi}{T}$ is the angle described per unit of time, that is the angular velocity. The angular velocity is often denoted by the Greek letter ω. Thus—

$$\omega=\frac{2\pi}{T}.$$

From this and the formula (3), we get

$$F=m\omega^2r \qquad (4)$$

as the expression in kinetic units for the centrifugal force with given angular velocity. The formula is in effect the same as (3).

If the angle turned through per second were given in degrees, it would be necessary, in order to make it suitable for the formula just given, to reduce it to radian measure. This is done by dividing the number that expresses the angular velocity in degrees per second by 57·3, as there are 57·3 degrees in a radian.

Example.—A stone weighing 2½ lbs. is attached to one end of a cord and whirled round, in a horizontal circle of three feet radius, twice per second. Find the centrifugal force.

Here

$$m=2{\cdot}5 \quad T=\tfrac{1}{2} \quad r=3,$$

hence

$$\omega=\frac{2\pi}{\frac{1}{2}}=4\pi=12{\cdot}57 \text{ radians per sec. nearly.}$$

$$F=2{\cdot}5\times(12{\cdot}57)^2\times3=1185 \text{ kinetic units, or } 36{\cdot}8 \text{ lbs.}$$

The student may exercise himself here by calculating the tension of the cord, and its inclination to the horizon, and its length.

60. The Pendulum.—The *simple pendulum* is, theoretically, a heavy particle suspended from a fixed point by a massless inextensible string, and vibrating freely in one plane. Practically a small leaden bullet suspended by a fine silk thread from a very firm support, and vibrating in one plane, realises the theoretical conception. The simple pendulum is so called in contradistinction to the *compound pendulum.* A rigid body of any shape, suspended so as to vibrate about a fixed axis, is called a compound pendulum; and thus the pendulum of a common clock, consisting of a rod, often made of wood, which carries a heavy bob, generally of brass or of mercury amalgam in a glass vessel,* is a compound pendulum. The simple pendulum would be unsuitable for the purposes that pendulums are used for in mechanism; but it is very important as a theoretical conception. The laws of motion of the simple pendulum can be investigated without much difficulty, and the results of the investigation can afterwards be applied to the case of the compound pendulum.

A is a small ball of lead suspended from a stout support at O by a very light silk thread. If left free, it would hang down with the thread in the line OC, the direction of the force of gravity at the place. If this ball be drawn aside to B, and then let go, care being taken not to give it any impulse to one side or the other of the line in which it is drawn aside from C, it will vibrate to and fro in one plane, viz., the vertical plane containing OC and B, and will evidently describe an arc of a circle whose radius is the length of the string.

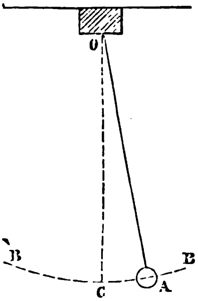

Fig. 64.

When the ball is hanging down in the middle position, its gravity is equilibrated (fig. 65) by the upward pull in the string. But when it is drawn aside from C to any position, such as B, the force of gravity, which is always vertically downwards, is not in the same straight line with the string, and thus is no longer equilibrated by the pull in the string. From figure 65 we see that the resultant of gravity, and the pull in the string, gives always a force directed in such a way as to make the ball return towards C. In drawing the ball from C back to B, it will be seen that the ball is raised a little above C. It tends to fall from B, down in the vertical line drawn through B. The string prevents it from doing so; but guided by the string from point to point it falls down to C along the curved path. During the fall to C it acquires velocity, and when it reaches C it is moving with this acquired velocity in the direction of the tangent to the circle at C, that is, horizontally. Guided by the string, it now commences to ascend the curve CB′, and were it not for resistance of the air, and loss of energy at the point of support, it would rise to B′ on the same level with B. There it would stop, commence to fall in again towards C, thence rise again to B, and so on for ever. Resistance of the air and loss of energy, due to want of perfect fixedness of the point of support, gradually bring it to rest; but a well-suspended simple pendulum will go on oscillating for a very long time. The distance from the point of suspension to the centre of gravity of the moving mass is called the "length of the pendulum;" the time of a swing from one end of the "range" to the other, is called the "time of vibration" of the pendulum.*

---

* The well known arrangement for a "compensating pendulum,"

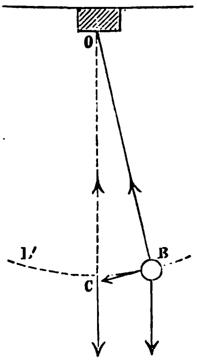

Fig. 65.

---

* This is not the case in any other kind of vibration. The time of vibration, or, as it is called, the "period" of the motion, is properly the time of making a *complete* oscillation, so that during the period the motion goes through *all* its phases. Hence the "period" of the pendulum is properly *the time from one end of the range to the other and back;* or, still better, it is the time that elapses from the instant when the moving mass passes through its middle position, till the instant when it next passes through the middle position going in the same direction as before.

The most important question regarding the pendulum is the time of vibration. Galileo first discovered that when the range is small, the *time of vibration is independent of the range.* Let a simple pendulum, such as we have been describing, and 39 in. long,* be caused to vibrate first through $\frac{1}{4}$ of an inch on each side, then through half an inch, then through an inch, then through two inches, and 100 vibrations counted each time with the aid of a clock or watch with a seconds hand, it will be found that the time of 100 vibrations is very closely the same in each case.†

The reason of this remarkable property, which gives to the pendulum its value for purposes of time-keeping, is thus explained. Although when the pendulum vibrates over the range BB′, the distance traversed is greater than when DD′ is the range, yet in the former case the pendulum falls from a greater height, viz., from B to C, than in the latter. The velocity acquired in falling from B down to any point P, between D and D′, is so much greater than that acquired in falling from D to the same point, that the additional velocity over the course DD′, in the case of the wider range, compensates for the increase of the range.

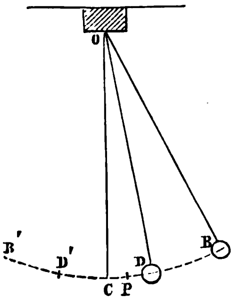

Fig. 66.

We have stated above that the time of oscillation is the same for different ranges *when the range is small.* The range is said to be small when the length of the arc through which the pendulum vibrates is a small proportion of the length of the pendulum. Thus with a pendulum

---

* A pendulum which beats seconds is about 39 inches long.

† Every student should try this instructive experiment for himself. A small bullet or stone hung from a nail by a fine *thread,* and a watch or clock, are all the apparatus required.

39 in. long, which would beat seconds, a range of half an inch, an inch, or two inches on each side may be reckoned small. When with such a pendulum the arc of vibration is 5, or 6, or 10 in. on each side, it can no longer be called small. But if the pendulum were 20 ft., or 30 ft. long, as it may be for illustration in a lecture room, 10 in. on each side would then be so small a range, that the difference between the time of vibration for that range and for a range of only 5 in. would not be noticeable, and could only be shown by a careful experiment, counting the vibrations for several minutes.* When the time of vibration of the simple pendulum is spoken of, what is commonly meant is the time of vibration calculated as though the range were infinitesimal. The formula for calculating it (5) below, is called the "pendulum formula;" and the time of vibration calculated by that formula agrees closely with the time observed by experiment, when the pendulum is vibrating over such short ranges as we have been speaking of. When the range is considerably greater than a $\frac{1}{20}$ or $\frac{1}{10}$ of the length of the pendulum, the vibrations are performed in periods sensibly longer than when it is very small. The correction to be applied for the purpose of calculating the time of vibration of a given pendulum vibrating over any range is not difficult, though we cannot enter into the question here.

The following are the laws of vibration of the simple pendulum.

(1.) For a given length, and at a given place on the earth, the time of vibration is not influenced either by the mass (quantity of matter) of the "bob," or by the material of which it is made. This law is the result of careful experiments. It is our best proof that the force of gravity on each body is proportional to its mass, and does not depend on the nature of its constitution. Were the force of gravity per unit of mass (Art. 37), more intense on some bodies than on others, different bodies similarly suspended would vibrate in different times, those that are most intensely actuated by gravity vibrating in shortest times. This is not the case.

---

* The range in the figures is unavoidably exaggerated.

(2.) The time of vibration of the simple pendulum depends on the length of the pendulum. For different pendulums at the same place on the earth, the *times of vibration are proportional to the square roots of the lengths.* Thus, of two pendulums, one four times as long as the other, the time of vibration of the first will be twice that of the second.

(3.) The same pendulum has not the same time of vibration at different places on the earth. This is on account of variations in the force of gravity. *The times of vibration of the same pendulum at different places are inversely proportional to the square roots of the numbers that express the intensities of gravity at those places.*

The last two laws are summed up in the formula for calculating the time of vibration of the simple pendulum. Let l be the length as defined above, let g be the number that expresses the force of gravity in kinetic units per unit mass at the place, and let t be the time of vibration, then the range being small

$$t=\pi\sqrt{\frac{l}{g}}. \qquad (5)$$

The letter π has its usual signification, viz., the ratio of the length of the circumference of a circle to the diameter, that is approximately the number 3·1416.

Examples.—(1.) Find the time of vibration of a pendulum 30 inches long at a place where gravity is 32·2.* Here $l=2·5$ feet

$$t=3·1416\times\sqrt{\frac{2·5}{32·2}}.$$
$$=·8752$$

or the time of vibration is ·875 of a second.

(2.) Find the length of the "seconds pendulum" (that is, a pendulum that beats seconds) at a place where gravity is 32·18. Here $t=1$; and thus—

---

* Note that g and l must have the same unit of length.

$$1 = 3{\cdot}1416 \times \sqrt{\frac{l}{32{\cdot}18}}.$$

Hence squaring, etc.,

$$l = \frac{32{\cdot}18}{(3{\cdot}1416)^2} = 3{\cdot}\dot{2}6 \text{ feet} = 39{\cdot}1 \text{ inches.}$$

61. The Compound Pendulum.—This is the name given in books on rigid dynamics, the advanced part of dynamics that deals with the motions of rigid bodies, to a rigid mass of any shape whatever, vibrating about a fixed axis. The general theory of the motion of such a body includes, of course, the motion of such pendulums as are used for time-keeping and in philosophical experiments, for the purpose of determining by observation the intensity of the force of gravity at various places on the earth's surface. Generally speaking, investigations regarding the motions of rigid bodies possess considerable complexity. In the present case, however, the results may be stated in a very simple way by comparison of the motion of the compound pendulum with that of the simple pendulum.

Let us consider the motion of a rigid body, such as a flat bar, AB, suspended so as to oscillate about an axis through C. Left to itself it would hang down with G, the centre of gravity, vertically under C, the point or line of support. But if it be drawn aside, as shown in the diagram, a couple, indicated by the lines of forces, acts on it to cause it to return to its normal position. Now, by considering the system of forces, and the conditions of the motion, it can be shown that the motion of the rigid body AB is precisely the same as it would be, if the mass of AB could be concentrated at a certain point

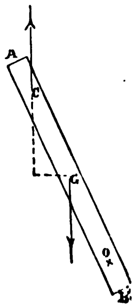

Fig. 67.

O, leaving the remainder of the body a massless frame, and if the rigid body thus formed could be suspended from C, and set to oscillate as a simple pendulum. The position of the point O, relatively to C and G, depends on the arrangement of mass of the rigid body; but we can always find a simple pendulum, the vibrations of which will have precisely the same period as those of any given rigid body vibrating about a given axis. Consequently, it is customary to speak of the *length of the equivalent simple pendulum* for a given compound pendulum. The distance CO is the length of the equivalent simple pendulum for the bar AB. The determination of the length of the equivalent simple pendulum for a given compound pendulum is a very important practical problem. This being determined, the time of oscillation for the compound pendulum is calculated by means of formula (5), p. 118.

The length of the equivalent simple pendulum for a compound pendulum of given form may be calculated from theory. In practice, however, it is determined by experiment. One way of doing this is, obviously, to compare the vibrations of the compound pendulum directly with those of a simple pendulum. Thus, let a small bullet be attached to a fine silk thread, and, by trial, let the length of the thread be adjusted till the vibrations of the simple pendulum agree with those of the compound pendulum. The length of the simple pendulum may then be measured.*

The best method of finding the length of the equivalent simple pendulum is, however, based on the following principle:—It can be shown that if the point O be found, such that a simple pendulum of length CO would keep time precisely with the compound pendulum AB vibrating about a given axis through C, then if the pendulum were suspended so as to vibrate about a parallel axis through the point O, the vibrations would be performed in precisely the same times. This principle may be stated in a slightly different form.

---

* The student should experiment for himself with a tolerably uniform lath, such as the laths used by Venetian-blind makers. A fine needle passed through the lath at C, fig. 67, will furnish the axis of suspension. Having found by trial the length CO of the equivalent simple pendulum, he must pass another needle through the point O thus determind, and prove the property of "convertibility" by experiment. By means of slips of lead, folded round and glued on to the lath, he can produce compound pendulums of varied distribution of mass.

A pendulum AB is given, vibrating about a given axis through C. From G, the centre of gravity, let a line be drawn at right angles to the given axis through C. The point where this line cuts the axis through C is called the *centre of suspension.* Let this line be produced in the direction of O to the point O, such that CO is equal to the length of the equivalent simple pendulum. The point O thus found is called the *centre of oscillation.*

The statement given above may now be put in the following form:—*The times of vibration of the pendulum about parallel axes, through the centres of suspension and oscillation are equal.* This is often spoken of as the *principle of convertibility of the centres of suspension and oscillation.*

62. Determination of the Force of Gravity. — The determination of the intensity of gravity at different parts of the earth's surface is, as we have seen (Chap. IV.), a problem of great importance. It is thus that the distribution of matter in the earth is found, and the figure of the earth ascertained.

If we could experiment with sufficient accuracy on falling bodies, we could deduce from the velocity of descent at various places the variations in the force of gravity. This, however, has not hitherto been found practicable. It is by means of pendulum experiments that the intensity of the force of gravity has been ascertained. We have seen that if g be the number that expresses the force of gravity on unit of mass,* if l be the length of a simple pendulum, and t the time of an oscillation, then

---

* The attention of the student is called to the fact that g is sometimes said to represent the number that expresses the force

$$t=\pi\sqrt{\frac{l}{g}}.$$

Hence it follows at once that

$$g=\frac{\pi^2}{t^2}\times l.$$

If then we measure the length l, and find by observation the time t of an oscillation, we can calculate g.

Example.—A pendulum 37·8 inches long is found to make 122 beats in two minutes at a certain place. Find the numerical value of g (the force of gravity in kinetic units per pound of matter) at that place.

Here, since 122 beats are made in 120 seconds, t the time of a single beat is $\frac{120}{122}$, and l, in feet, being $\frac{37 \cdot 8}{12}$, we have (putting for π its value 3·142)

$$\begin{aligned} g&=\frac{(3\cdot142)^2}{\left(\frac{120}{122}\right)^2}\times\frac{37\cdot8}{12} \\ &=(3\cdot142)^2\times\left(\frac{122}{120}\right)^2\times\frac{37\cdot8}{12} \\ &=32\cdot14. \end{aligned}$$

In 1818, Captain Kater first took advantage of the principle of convertibility of the centres of suspension and oscillation to obtain very accurate determinations of the intensity of gravity by observations with the compound pendulum. AB (fig. 68) is a thin bar of hard well-seasoned wood. W and w are two weights, of which W is much the greater. The weight w is movable, and can be clamped in any position by a screw. C and O are two steel knife edges, and the pendulum can be made to vibrate about either of them, the knife edge resting on agate plates. To experiment with this pendulum, it is set

---

of gravity per unit of mass in kinetic units, and sometimes (*e.g.*, p. 101) the acceleration of the mass due to gravity. The fact is, that, when the kinetic system of measuring forces is employed, the numerical expressions for the acceleration and for the force *per unit of mass* are identical (see (4), p. 106).

to vibrate first about C, and then about O, and the vibrations are counted. If it does not vibrate in equal times about C and about O, the movable weight *w* is shifted; and, by trial, the position is found for *w*, which makes the times of vibration about C and O equal. When this is the case, the distance CO is the length of the equivalent simple pendulum, according to the principle stated above, and this length is used in the formula for calculating the intensity of gravity. The time *t* of vibration about either C or O is then ascertained with great accuracy by counting the vibrations for a long time. The best way of doing this is by means of the "principle of coincidences," which is briefly as follows:—The experimental pendulum, and the pendulum of an astronomical clock, are placed so that, when both are hanging vertically, one is hidden by the other to an observer, who, usually with a telescope, looks at them from a distance. Suppose the experimental pendulum to vibrate a very little faster than the pendulum of the clock, which we may suppose beats seconds exactly, and that both are started together. Then on the first beat both pendulums will cross the field of view of the telescope together, but after that the experimental pendulum will pass across the field of view in advance of the clock pendulum, and the difference between the times of passage will go on increasing until at last both pendulums again cross the field of view together, the experimental pendulum having gained one complete vibration (two beats) on the clock pendulum. Suppose this takes place in 1000 seconds as indicated by the clock, then the time of vibration of the experimental pendulum is plainly $\frac{1000}{1001}$

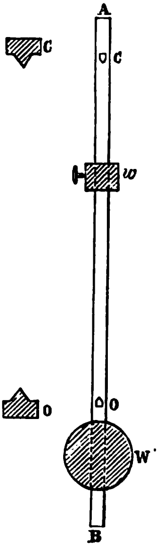

Fig. 68.

of a second. This method is found to give results with great accuracy. It was employed by Kater, Sabine, Airy, and others in their experiments on gravity. It is by pendulum experiments that the results as to variations of gravity over the earth's surface referred to in Art. 37 have been obtained.

63. Projectiles.—We may now very briefly consider the motion of a body projected obliquely near the surface of the earth, and under the influence of gravity. Let us leave out of account for the present the resistance of the air, and let us consider a body projected from A in the direction AA′, and with a given velocity. According to Newton's laws, Art. 19, if the body were not under the influence of any force, it would continue to move uniformly in the line AA′, passing over equal spaces in equal times. But under the action of gravity it is drawn from the line AA′, and describes a curved path. Let us consider the motion of the body resolved into two components vertical and horizontal. Now, since the direction of gravity is always vertical, it is plain that the horizontal component of the motion will remain unchanged, and that only the vertical component will be altered. Let a_1, a_2, etc., be the points that the body would have reached at the ends of 1st, 2nd, etc., seconds had there been no gravity

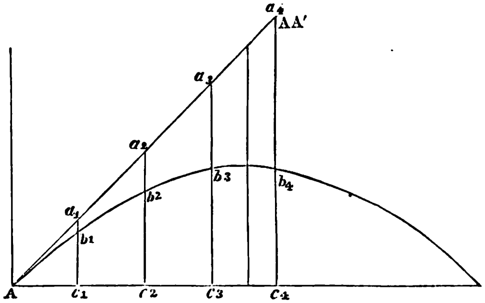

Fig. 69.

acting on it; and let vertical lines be drawn through a_1, a_2, etc. Since the horizontal component of the velocity of projection remains unaltered, the body will be found as far from A in the horizontal direction as if gravity were not acting; that is, at the end of the first second it will be somewhere in the vertical a_1c_1, at the end of the 2nd second it will be somewhere in a_2c_2, and so on. But owing to gravity the body does not rise so high in the vertical direction as it would were gravity not in action, and thus at the ends of the successive seconds, it is found below the points a_1, a_2, etc. The simplest way of tracing out the path of the projectile is to mark off, according to the scale on which Aa_1, a_1a_2, etc., are laid down, points b_1, b_2 on the verticals a_1c_1, a_2c_2, etc., such that the distance a_1b_1 is the space that a body falling freely would fall in one second, that is 16 feet, a_2b_2 that fallen in two seconds, and so on according to the formula 2_a, p. 103. It will readily be seen that the points b_1, b_2, are those at which the body must be at the ends of the corresponding seconds. Drawing with the hand a smooth curve through Ab_1b_2, etc., the path is found. It is a portion of a parabola with the axis vertical. When the velocity of projection is small, it will be necessary to take the points a_1, a_2 at shorter distances than correspond to successive seconds, in order to get a sufficient number of points on the curve. Intervals of a quarter or a tenth of a second may then be taken, but the principle is, of course, the same as that which has been illustrated. In the diagram, fig. 69, the velocity of projection is taken as being 160 feet per second. The scale of the diagram is $\frac{1}{20}$ in. to 16 feet.

When the resistance of the air is taken into account, the problem of the motion of projectiles becomes very complicated, and, indeed, owing to our want of knowledge of the laws of the resistance of the air, it has not been worked out mathematically. Observation, however, tells us that the form of the *trajectory* of a body like a round cannon ball is very far from being parabolic. The diagram, fig. 70, shows a trajectory such as may be seen when a large ball is fired from a point A, with an elevation of about 33 degrees.

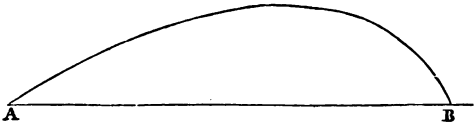

Fig. 70.

CHAPTER VIII.

WORK—ENERGY.

64. Work.—When a body moves, or is moved, against resistance from one position to another, resistance being thus overcome along the path, *work* is said to be done. In the first case the body is said to do work; in the second case the agent by which the body is moved does work upon the body, and work is expended by the body in overcoming the resistance. When a bullet strikes a wooden target and penetrates it, the bullet does work against the resistance opposed to its passage. When a heavy body is raised from the earth, work is done in overcoming the attraction of the earth. Again, work is done when we press against a spring that yields before the pressure, or when we draw asunder two magnets that are attracting each other. To constitute work force must be in action, and the action must take place over a path or distance. Thus we do not speak of work being done while the heavy body which has been raised from the earth is held supported, or while the spring that has been bent is held bent, or while the magnets that have been separated are held apart.

When the resisting force is constant over the path, the *work done is measured by the product of the magnitude of the force, and the length of the path.* The most common unit of work is the *foot-pound.* It is the amount of work required to raise one pound mass through one foot against the force of gravity. When a weight of one pound is raised 10 ft. one foot-pound of work is done in lifting it through each foot of the path. Again, when a weight of 6 lbs. is raised one foot, one foot-pound of work is done for each pound raised. Thus, when a weight of 6 lbs. is raised 10 ft., the work done is 6 × 10, or 60 foot-pounds.

The foot-pound is a *gravitation* unit, and, as it depends on the gravitation unit of force, it is slightly different in amount at different places on the earth's surface. Thus, while it is eminently convenient for the practical measurements of engineers, it is replaced for scientific measurement, in most departments of physical science, by the absolute or kinetic unit of work, which is the work done when one kinetic unit of force is overcome through one unit of length. Thus, when the British system of measurement is employed, work is measured in British kinetic units of force acted against through a distance of one foot, that is, in foot-poundals (Art. 22). When the metrical system is used the unit of work is one dyne overcome through one centimetre, and is called the centimetre-dyne. *To reduce foot-pounds to foot-poundals, we must multiply by* 32·2; *to reduce foot-poundals to foot-pounds, divide by* 32·2. The factor for reduction on the metrical system from centimetre-grammes to centimetre-dynes, and *vice versa,* is 981·4. This will be readily understood from the explanations given in Art. 22. The absolute unit on the metrical system is employed for all measurements regarding work done by electric currents, by the motion of magnets, etc. A special name, the *Erg* (from *ergon,* the Greek for work), is given to the centimetre-dyne.

When a weight is raised obliquely, as along a smooth inclined plane, the resistance opposed to the force which raises the weight is not the whole gravity of the body.

It is only that component of it which acts along the plane that has to be overcome (Art. 45); and it is this component force which must be used in the calculation of the work done. In every case where the force is oblique to the path of the body, the work done is measured by the product of the *component force in the direction of motion and the length of the path.*

65. Rate of doing Work. — The rate at which a machine or other agent can perform work often requires to be considered. It is measured by the amount of work that the machine or agent can perform per unit of time. For specifying the rate at which machines can do work, Watt's practical unit, the *horse-power*, is very generally adopted. An engine that can do 33,000 foot-pounds of work per minute is said to be of one horse-power. An engine that can do double that amount of work per minute is said to be of two horse-power, and so on. The unit is founded on an estimate made by Watt of the amount of work that horses perform. It is found that the estimate was, in reality, much above the power of an ordinary horse.

66. Energy of a Body or System.—Energy of a body or system is the capability that the body or system possesses of doing work, and when a body is giving out work the work done is often spoken of as energy expended. There are various forms in which bodies may possess energy. For example, the spring of a watch when wound up possesses energy, in virtue of which it keeps the mechanism going against friction, thus doing work while it unwinds; and when it has "run down," it has lost the energy that it possessed before. The weight of a pile-driver just before it strikes the pile possesses energy in virtue of the velocity with which it is moving; so also does a bullet that has been fired from a gun. There are also many other forms in which energy exists. Thus, gunpowder or nitro-glycerine possesses a latent energy due to the chemical arrangement of its ingredients; steam possesses energy in the heat it contains; an electric current possesses energy in virtue of electro-magnetic inertia. These are given as examples, but we cannot attempt to discuss the subject fully here.

Energy, as known to us, belongs to one or other of two great classes, to which the names *potential energy* and *kinetic energy* are given. Potential energy is energy that a body possesses in virtue of its position, or that a system possesses in virtue of the relative positions of its parts. Kinetic energy is energy that a body possesses in virtue of its being in motion. As examples of potential energy, we may consider the energy of a weight raised up above the earth, as the weight of a clock, which does work in falling; the energy of a bent watch spring, which does work in uncoiling; the energy of compressed air, as in an air-gun, or in an air-brake on a locomotive, which does work in expanding; and, again, the energy of nitroglycerine, or of a mixture of oxygen and hydrogen, due to chemical separation, and in virtue of which work is done, while the chemicals combine and change their relative arrangement. Examples of kinetic energy are presented to us in the energy of a fly-wheel revolving rapidly, in the energy of a cannon bullet in motion, or in the energy possessed by the weight of a piling engine just before it strikes the pile. It is in virtue of its motion that the weight of the piling engine can produce the effect that it does produce. It might lie at rest on the head of the pile for ever without forcing the pile into the ground. The energy of a heated body is another kind of kinetic energy, though this cannot be explained properly here.

It has been said that the energy of a body or system is its capability of doing work. This statement must, in our present state of knowledge, be taken as a relative one. Just as when the velocity of a body is specified relatively to some other body considered for the time being as at rest, so it is with the energy of a body or of a system of bodies. When we speak of the velocity of a ball rolling across the deck of a ship, we leave out of account the motion of the ship along the surface of the earth. Going further, we have to remark that the ship is in motion carrying the ball with it; the surface of the earth has a motion relatively to the axis of the earth; while the earth itself is being carried round the sun once in a year. Then we have to take into account the motion of the solar system with respect to other systems, and so on. We know nothing whatever of any point in space as absolutely fixed. So it is with respect to energy. We know nothing of an absolute zero of energy, or of circumstances in which we could consider any body or system to have parted with all its energy so as to be incapable of giving out or producing any more work. To take a simple example: When the weight of a clock is wound up, a certain amount of work is done upon it, in virtue of which it acquires the power of keeping the clock in motion till the weight has run down. But this is not all the energy that the weight possesses. If the clock were placed at the edge of a precipice, or of a deep pit, the weight could do more work by falling down further. Even could it be allowed to fall in to the centre of the earth, it would still possess energy in virtue of its position with respect to the sun, and with respect to the other attracting bodies in the universe. Then when we come to take into account the energies due to temperature, due to electrification, due to magnetization, due to chemical affinity, that may all co-exist in the same body, the question becomes still more complicated. These relations, however, even so far as they are known to us at present, cannot be discussed fully here, as they involve questions in Natural Philosophy and Chemistry that are beyond the scope of an elementary treatise on Dynamics.

67. In conclusion, we will consider the kinetic energy of a moving mass, that is, the energy that it possesses in virtue of its motion. A moving mass, such as a cannon ball, does work when it strikes against a resisting object, and in doing so parts with its velocity and comes to rest. *The kinetic energy of a moving mass is measured by the* *mechanical work that it can perform or give out in parting with its motion.*

Let a body, whose mass is m lbs., be moving with a velocity v feet per second; and let the direction of motion be vertically upward. According to Art. 53, p. 104, the height h in feet to which it will rise, when resisted only by gravity,* is given by the formula

$$v^2=2gh.$$

The body thus rises h feet vertically, doing work against the force of gravity. Let us now multiply each side of this equation by m and divide by 2. Thus—

$$\frac{mv^2}{2}=m.g\times h.$$

But mg is the force of gravity expressed in kinetic units (poundals) acting against the body during its rise, and h is the number of feet of height or of length of the path through which the force was acting during the time that the body was coming to rest. The product $mg\times h$ (that is, the number of units of force multiplied by the number of units of length) is, according to Art. 64, the mechanical work that the body was able to do in being deprived of its velocity. Thus, since $mgh=\frac{mv^2}{2}$,—*The kinetic energy of a mass m moving with velocity v, is* $\frac{mv^2}{2}$ *in kinetic units, and* (according to Art. 22) *is* $\frac{mv^2}{2g}$ *in gravitation units.*†

In the case that we have just been considering where a body, in virtue of kinetic energy that it possesses, rises against gravity to a height above the earth's surface, the kinetic energy of the moving mass is converted into potential energy. The mass thus raised up acquires—in virtue of its position relatively to the earth—energy, in virtue of which it can perform work in falling again to the earth's surface. Thus, if by some arrangement the body is caught at the highest point of its course, it can be made to do work in falling. It can be set to drive a clock or other machinery for example. *The potential energy of a body raised up from the earth's surface is equal to the work required to bring it from the earth's surface to the place it occupies.* Thus a body whose mass is m lbs., raised to a height h feet, possesses mgh kinetic units (foot-poundals), or mh gravitation units (foot-pounds) of potential energy.

---

* Here resistance of the air is supposed to produce no retardation, and thus gravity is the only force that we have to consider.

† The name *vis viva*, which is now all but obsolete, having been replaced by the expression *kinetic energy of a moving mass*, was given by Leibnitz to the product mv^2, which, it will be seen, is twice the numerical value of the kinetic energy of the mass m moving with velocity v.

Finally, let the student observe that since the numerical values of $\frac{mv^2}{2}$ and mgh are identical for the case that we have been considering, the values of the kinetic energy and of the potential energy into which it is convertible are the same. He will find it easy to prove in the converse case where a body, whose mass is m pounds, is let fall from a height h feet, that the potential energy of the mass raised from the earth's surface is equal to the kinetic energy that the mass acquires in falling to the surface (the resistance of the air being considered as nothing). This is a particular case of the grand Principle of Conservation of Energy.

APPENDIX.

TABLES FOR COMPARISON

OF

BRITISH AND METRICAL MEASURES.

1 inch	2·5400 centimetre.
1 foot	30·4797 centimetre.
1 statute mile	1·609 kilometre.
1 nautical mile	1·852 kilometre.
1 metre	39·370 inch.
1 metre	3·2809 feet.
1 kilometre	0·6214 statute mile.

By Captain Clarke's determination (1866), one metre is 39·370432 inches, the Standard Metre being taken as correct at 0° Centigrade, and the Standard Yard at 16⅔° C.

Surface.

1 square inch	6·5416 square centim.
1 square metre	10·764 square feet.

Capacity.

1 cubic inch	16·387 cubic centim.
1 cubic foot	28·316 cubic decimetre (litre).
1 pint	·5676 litre.
1 litre (1000 cubic centims.)	61·025 cubic inch.

Weight.

1 grain	64·799 milligramme.
1 pound avoirdupois	453·59 gramme.
1 ton	1·016 metrical tonne.
1 gramme	15·4323 grain.
1 kilogramme	2·2046 lb.
1 tonne (1000 kilos.)	·98421 ton.

By the determination of Professor W. H. Miller (1844), the Standard "Kilogramme des Archives" is 15432·34874 grains, of which the Standard Pound contains 7000.

EXERCISES AND EXAMINATION QUESTIONS.

(*A considerable number of the following exercises are selected from the Examination Papers for Science Schools and Classes, published by the Science and Art Department. Answers, instructions, and references to aid the student in answering the questions are included in square brackets* [ ].)

In numerical calculations it ought to be a rule that the answer shall not be given with a long and useless string of figures, pretending to exactitude altogether beyond that warranted by the physical data; but in making approximations the result must be worked out in such a way that the *last* "significant figure," presented in the answer shall make the answer the nearest to that which would be obtained were it quite fully worked out from the data offered.

1. Explain what is meant by the composition of velocities. Find by construction or otherwise the resultant of two uniform velocities of 12 feet per second and 17 feet per second, the directions being inclined at 60°.

2. A river is flowing down a straight channel at one mile per hour. A boat is rowed across it, and kept always at right angles to the line of the channel, and is thus, while it crosses, carried downward with the river. The boat is rowed at 4 miles per hour. Find the direction of motion of the boat relatively to the land.

3. A river is flowing at half a mile per hour down a straight channel. A boat is rowed across, and steered so as to point 18° up stream. How fast must it be rowed that it may cross at right angles to the stream? If the river is 200 feet wide, how long will it take to cross? [This exercise will be performed most easily by construction.]

4. The velocity of a train is known to have been increasing uniformly. At 1 o'clock its velocity was 12 miles an hour, at 10 minutes past 1 o'clock its velocity was 36 miles an hour. What was its velocity 7½ minutes past 1 o'clock?

5. State Newton's first law of motion, and explain what is meant by *inertia* of matter.

6. Mention the points that must be specified that a force may

be considered to be known. What is meant by a force of one pound?

7. Define the absolute or kinetic unit; and specify the kinetic units on the British System and on the Metrical System [Arts. 20, 22].

8. Reduce 6 pounds force at London to British kinetic units. Express the weight or gravity of 24 grammes at London in *dynes.* (Force of gravity per pound of matter is 32·191 poundals at London at sea-level or 981·16 dynes.) [Art. 22.]

9. Express 450 poundals in pounds force at London.

10. (α.) State exactly what is meant when it is said that the accelerating force of gravity near the earth's surface is very nearly 32·2 in feet and seconds. (β.) A body falls freely under the action of gravity from rest for 6 seconds, what is the space described in the last 2 seconds of its motion? [(α.) The expression "accelerating force of gravity" is an old name for what is better called the *acceleration* due to gravity, or else it is the intensity of gravity per unit of mass. The acceleration due to gravity is the acceleration that a body is subject to when it is falling freely, or the velocity in feet per second gained per second by a body falling freely. As a body falling freely from rest for one second acquires a velocity of 32·2 feet per second, the number 32·2 represents the acceleration due to the force of gravity (Arts. 15 and 18). The number 32·2 also expresses the force of gravity in British kinetic units per pound of matter (Art. 37, p. 69). (β.) The formula for calculating the space fallen from rest in any time, t is

$$s = \tfrac{1}{2} gt^2.$$

$$\text{If } t=6 \quad s = \tfrac{1}{2}\, 32{\cdot}2 \times 36$$

$$t=4 \quad s' = \tfrac{1}{2}\, 32{\cdot}2 \times 16.$$

The difference or space fallen in the last two seconds

$$= s - s' = 16{\cdot}1 \times 20 = 322 \text{ feet.}]$$

11. A body under the action of a constant force describes in three successive seconds distances of 12 ft., 18 ft., and 24 ft. respectively; what ratio does the force per unit of mass producing the motion bear to the force of gravity per unit of mass? ($g=32$.) [Here the increment of velocity, or the acceleration, is 6 feet per second gained per second. But the acceleration is numerically equal to the force per unit of mass. Thus the force per unit of mass is 6 kinetic units (poundals). The intensity of gravity per unit of mass is taken to be 32 kinetic units (poundals). Thus the ratio required is 6 : 32.]

12. A body whose mass is 8 lbs. is known to be under the action of a single constant force. It is observed to move from

rest, and in the first second of its motion to describe a distance of 5 ft. What is the magnitude of the constant force? [See p. 106 (4) and (6).]

13. How long will it take a body to fall freely from rest through a vertical height of 192 yards? ($g=32.$) [192 yards = 576 feet. Take the square root of 576 and divide the result by 4 to obtain the time in seconds. See Art. 54.]

14. A body moves from rest under the action of a constant force; how does it appear that the spaces described in successive seconds form an arithmetical progression? [To answer this question apply the method of the example, Art. 55, to formula (6), p. 106.]

15. A body is projected vertically upward with a velocity of 80 feet per second. Find how long it will continue to rise. What will be its velocity at the end of three seconds, and in what direction will it be moving?

16. A stone is thrown vertically upward with a velocity of 200 feet per second. Find how high it will rise, and how long it will be before it returns to the point of projection.

17. A stone is thrown horizontally from the top of a tower 200 feet high with a velocity of 56 feet per second. Find where it will strike the ground. [Find first how long it will take to fall to the ground; and then find how far out from the tower it will have moved in that time with the given horizontal velocity.]

18. In an Attwood's machine the boxes hung at the ends of the thread are, each of them, 10 ounces; the equivalent for the inertia of the wheelwork is 3 ounces. What additional weight must be placed on one of them that the system may have an acceleration of 2 ft. per second, per second?

19. A plummet is hanging by its cord at a place neither on the equator nor at either pole. It is carried round with the earth's diurnal rotation. Show by a diagram in what way the direction of the plummet cord, and the amount of the force that it has to sustain, are influenced by the diurnal rotation.

20. Suppose that the earth could be made to rotate on its own axis in 6 hours instead of 24, and could still retain its present form, what would be the gravity of a 20 lb. weight at the equator? Take the earth's radius as 21×10^6 (21,000,000) feet.

21. A rod, whose weight can be neglected, rests on two points 12 inches apart; a weight of 10 lbs. hangs on the rod between the points, and 4 inches from one of them. What is the pressure on each point? [The pressures on the points of support are *inversely* as the distances of the weight from those points].

22. Three particles A, B, C, are placed so as to form the corners of a triangle right angled at B. The mass of A is 6 lbs., that of B 10 lbs., that of C 9 lbs. The distance AB is 3 feet; and the distance BC is 2½ feet. Find the centre of inertia.

23. Draw two straight lines, AB, AC, containing an angle of 60°. Let a force of 12 lbs. act from A to B, and one of 23 lbs. from A to C. Find by construction, or otherwise, the magnitude of their resultant.

24. A ball weighing 3 lbs. is hung up by a cord, and is drawn aside from the normal position of equilibrium by a cord pulling horizontally, as in fig. 6, p. 28. The inclination of the cord AB to the horizon is 18°. Find by construction, or otherwise, the force applied by the horizontal cord and the pull in cord AB.

25. Draw an equilateral triangle ABC; suppose a force of 10 lbs. to act from A to B, another force of 10 lbs. to act from B to C, and a third force of 15 lbs. from A to C; find their resultant, and show by a diagram exactly how it acts.

26. State exactly what is meant by a body being (1) in stable (2) in unstable equilibrium. Illustrate your answer by the case of a loaded sphere placed on a horizontal plane. [Art. 49.]

27. Describe briefly a common steelyard. How does it appear that the graduations are equal?

28. A uniform cube is placed on a rough table with its edges parallel to the edges of the table. The table is then tilted at one end till the cube, which does not slip at all, topples over. At what inclination does this happen?

29. A loaded cylinder, such as A, fig. 58, p. 93, rests on an inclined plane with its axis horizontal; show by a figure the greatest inclination that the plane can have, placing the centre of gravity of the cylinder at any distance you choose from the centre of the cylinder.

30. If a small heavy ball is suspended by a fine thread, 12 ft. long, find how many small oscillations it will make in one minute.

31. A pendulum which vibrates seconds of mean time in the latitude of London, at the level of the sea, and in a vacuum, measures 39·1393 inches, find the intensity of gravity per pound of matter there.

32. A body whose mass is 5 lbs. moves with a uniform velocity of 80 ft. per second in a circle whose radius is 10 ft.; what force must act upon it, and in what direction? In what units is your answer estimated?

33. Suppose the moon to revolve round the earth in a circle

uniformly once in 27¼ days, the earth being perfectly fixed. Find the total attraction between earth and moon.

Moon's mass approximately $6{\cdot}98 \times 10^{19}$ tons.
Moon's distance from earth approximately 240,000 miles.

34. (α.) State exactly what is meant by the work accumulated in, or *vis viva* of, a moving mass? (β.) A railway truck weighs 12 tons. It is drawn from rest by a horse through a distance of 50 feet, and is then moving at the rate of 3 miles per hour. If the resistances are 8 lbs. per ton, how many units of work must the horse have done on the truck? [(α.) "Accumulated work," or "*vis viva*," are old expressions for twice the kinetic energy of a moving mass. If a mass m is moving with a velocity v, then mv^2 is the *vis viva*, and $\frac{mv^2}{2}$ is the kinetic energy of the body (Art. 66). (β.) The work done by the horse is of two kinds. First, there is the resistance due to friction, etc., overcome through 50 feet, which is equal to $12 \times 8 \times 50$ foot-pounds, or 4800 foot-pounds. Secondly, there is the work done in imparting kinetic energy to the truck, since the truck is left moving with a velocity of 3 miles per hour. This is calculated by the formula $\frac{mv^2}{2g}$. Reducing m to pounds, and v to feet per second, and taking $g = 32$, we find this part of the work equal to

$$\frac{12 \times 2240\,(4{\cdot}4)^2}{2 \times 32} = 8131{\cdot}2 \text{ foot-pounds.}$$

Adding together the two parts of the work done—

Total work done $= 12931$ foot-pounds.]

35. A body weighing 50 lbs. is moved from a state of rest, and is found after a certain time to have a velocity of 10 feet per second. How many units of work must the forces causing the motion have done over and above those expended on the resistances? (take $g = 32$).

36. A weight of 3 tons is raised through 50 feet.:—(*a*) How many units of work (foot-pounds) must be done by the agent? (*b*) If the weight is raised in a quarter of a minute, what must be the horse-power of the agent?

37. A body weighing 100 lbs. moves without rotation at the rate of 20 miles per hour. Find its kinetic energy. If, from the instant under consideration, the body slides along a rough horizontal plane, find how far it will go before coming to rest, the co-efficient of friction between the body and the plane being 0·05. [*Here* the velocity reduced to feet per second is $\frac{20 \times 5280}{60 \times 60} = 29{\cdot}33$.

$$\text{Hence kinetic energy} = \frac{100 \times (29 \cdot 33)^2}{2} = 43022 \text{ foot-poundals,}$$

$$\text{Or} = \frac{100 \times (29 \cdot 33)^2}{2 \times 32 \cdot 2} = 1336 \text{ foot-pounds.}$$

To answer the second part of the question, it must be understood that *the co-efficient of friction is*, practically, *a number which, multiplied by the pressure on the plane, gives the retarding force due to friction.* The retarding force due to friction is found to be in simple proportion to the pressure on the plane. Thus the retarding force $= 100 \times \cdot 05 = 5$ pounds force. Now this retarding force, acting through the distance that the body runs before stopping, destroys all the kinetic energy that the body possesses; or the body, in virtue of the kinetic energy it possesses, is able to do work equal to 1336 foot-pounds against the retarding force of 5 pounds. Let s be the distance run in feet. Then $5 \times s$ is the work done over the distance. Hence

$$5s = 1336,$$
$$\text{or} \quad s = 267 \cdot 2 \text{ feet.}$$

The student will not fail to observe that the distance travelled is quite independent of the mass, since the kinetic energy possessed and the retarding force are, both of them, in simple proportion to the mass.]

38. How many gallons of water would a steam engine of 10 horse-power raise from a depth of 200 fathoms in an hour? (A gallon of water weighs 18 lbs.)

39. The modulus of elasticity of a substance is 30,000,000 (in lbs. per square inch), under what tension will a rod 30 feet long and 5 square inches in section be stretched ¼ of an inch? [See pp. 58 and 59.]

INDEX.

The figures refer to the respective Paragraphs.

Absolute or Kinetic system of measuring forces, 22.
Acceleration, 15.
Angular velocity, 59.
Atoms, size of, 38.
Attwood's machine, 51.

Balance, 42.

Centre of gravity, 38.
Centre of inertia, 39.
Centre of parallel forces, 31.
Centres of inertia of simple figures, 39, pp. 77, 78.
Centres of suspension and oscillation, convertibility of, 61.
Centreward force in uniform circular motion, 58.
Centrifugal force not an appropriate name, 57.
Centrifugal force, 58.
Centrobaric bodies, 36, 38.
Component forces, 24.
,, velocities, 13.
Composition of forces, 24.
,, velocities, 13.
Compound pendulum, 61.
Compound pendulum; length of equivalent simple pendulum, 61
Compressibility of liquids, 35.
Convertibility of centres of suspension and oscillation, 61.
Couple, arm of, 29.
,, moment of, 29.
Couples, 29.
Curvilinear motion, 58.

Definition of a fluid, 32, 34.
Derived units, Note, p. 14.
Determination of force of gravity, 62.
Diagrams of forces, 24.
Divisibility of matter, 33.
Dyne, 22.

Elasticity, modulus of, 34, p. 59.
Elements for specification of a force, 20.
Energy of a body or system, 66.
Energy of different kinds, 66.
Equilibrant, 24.
Equilibrium of body resting on a horizontal plane, 49.
Equality of two masses defined, 6
Erg, 64.

Falling bodies, formulas for the motion of, 53.
Falling bodies, laws of, 50.
,, ,, Galileo's experiments on, 50.
Foot-pound, 64.
Foot-poundal, 64.
Force, moment of, 30.
Force of gravity, determination of, 62.
Forces, graphical representation of, 23.
Formulas for the motion of falling bodies, 53.
French system of weights and measures, 4, 6, and Appendix.
Fundamental Units, Note, p. 14.

Galileo's experiments on falling bodies, 50.
Galileo's inclined plane, 50.
Gramme defined, 6.
Graphic representation of motion, 11.
Graphic representation of forces, 23.
Gravitation, 36.
Gravitation not uniform over earth's surface, 37.
Gravitation system of measuring forces, 21.
Gravity affected by centrifugal force, 37.
Gravity, centre of, 38.
Gravity, terrestrial, 37.
Guinea and feather experiment, 50.

Hooke's law, 34.
Horse-power, 65.

Ice, a viscous substance, 32.
Inclined plane, 45.
Inertia, centre of, 39.
Inertia of matter, 6, 16.

Kater's pendulum, 62.
Kinematics, 15.
Kinetic energy, 66.
Kinetic or Absolute system of measuring forces, 22.

Kinetic unit of force, 22.
Kinetic energy of a moving mass calculated, 67.
Kinetics, 17.

Laws of gravitational attraction, 36, p. 65.
Laws of motion, 19.
Length, standards of, 3.
Levers, 41.

Mass, standards of, 5, 6.
Matter, inertia of, 19, 6.
  ,, states of, 32.
Mechanical powers, 40.
Metre, Definition, of, 4.
Metric system, 4, and Appendix.
Mile, nautical, Note, p. 12.
Modulus of elasticity defined, 34, p. 59.
Moment of a force, 30.
  ,, couple, 29.
Moments, principle of, 30.
Momentum, 16.
Morin's machine, 52.
Motion, quantity of, defined, 16.
Motion represented graphically, 11.

Nautical Mile, Note, p. 12.
Newton's laws of motion, 19.

Œrsted's apparatus, 35.

Parallel forces, 28.
  ,, centre of, 31.
Parallelogram of velocities, 13.
  ,, forces, 25.
Pendulum, 60-61.
Period of an oscillating motion, p. 115, Note.
Plasticity of matter, 32.
Polygon of forces, 27.
Potential energy, 66.
Poundal, 22.
Power, mechanical, 40.
Preservation and restoration of standards of length and mass, 3, 4, 6.
Principle of moments, 30.
Projectiles, 63.
Pulley, 47

Quantity of Matter, 5.

Radian (59), p. 112.
Rate of doing work, 65.
Rectilinear motion of body under constant force, 55.
Resolution of velocities, 14.
  ,, forces, 26, 14.
Resultant velocity, 13.
  ,, force, 24.
Rigidity, 34.

Screw, 46.
Sidereal day, alteration of length, 2.
Simple pendulum, 60.
Size of atoms, 33.
Space described by point moving with uniform velocity, 8.
Space or length, standards of, 3.
Specification of a force, 20.
Stable equilibrium, 49.
Standard of time, 2.
Standards, national, preservation and restoration of, 3, 4, 6.
States of matter, 32.
Statics, 17.
Steelyard, 43.
Strain defined and measured, 34.
Stress, 34.

Terrestrial gravity, 37.
Time, standard of, 2.
Time of vibration of simple pendulum (5) p. 118.
Triangle of forces, 27.

Uniform circular motion, 58.
Units, fundamental and derived, Note, p. 14.
Units of force, Absolute or Kinetic, 22.
  ,, Gravitation, 21.
Unstable equilibrium, 49.

Variation of terrestrial gravity with the latitude, 37.
Velocity, 8.
  ,, angular, 59.
Velocities, composition of, 13.
  ,, resolution of, 14.
Vibration of simple pendulum, time of (5), p. 118.

Viscous yielding of semi-fluids, 32.
Vis viva, Note, p. 131.

Water, compressibility of, 35.
Wedge, 46.
Wheel and axle, 44.
Work, 64.

WILLIAM COLLINS AND COMPANY, PRINTERS, GLASGOW.

William Collins, Sons, & Co.'s Educational Works.

COLLINS' SERIES OF FIRST-CLASS SCHOOL ATLASES.

Carefully Constructed and Engraved from the best and latest Authorities, and Beautifully Printed in Colours, on Superfine Cream Wove Paper.

MODERN GEOGRAPHY—Crown Series.

s. d.

MY FIRST ATLAS, consisting of 12 Maps, 9 inches by 7½ inches, folded 8vo, in Neat Wrapper, ... 0 6

1 The Hemispheres.
2 Europe.
3 Asia.
4 Africa.
5 North America.
6 South America.
7 England and Wales.
8 Scotland.
9 Ireland.
10 Central Europe.
11 Australia.
12 Palestine.

THE PRIMARY ATLAS, consisting of 16 Maps, 9 inches by 7½ inches, 4to, Stiff Wrapper, ... 0 6

1 The Hemispheres.
2 Europe.
3 Asia.
4 Africa.
5 North America.
6 South America.
7 England and Wales.
8 Scotland.
9 Ireland.
10 Central Europe.
11 India.
12 Canada.
13 United States.
14 Australia.
15 New Zealand.
16 Palestine.

THE POCKET ATLAS, consisting of 16 Maps, folded in 8vo, and mounted on Guards, cloth lettered, ... 1 0

THE JUNIOR, OR YOUNG CHILD'S ATLAS, consisting of 16 Maps, 4to, with 16 pp. of Questions on the Maps, in Neat Wrapper, 1 0

THE SCHOOL BOARD ATLAS, consisting of 24 Maps, Crown 4to, cloth limp, ... 1 0

THE PROGRESSIVE ATLAS, consisting of 32 Maps, 9 inches by 7½ inches, 4to, cloth lettered, ... 2 0

1 The Hemispheres.
2 The World, (Mercator's Projection.)
3 Europe.
4 Asia.
5 Africa.
6 North America.
7 South America.
8 England and Wales.
9 Scotland.
10 Ireland.
11 France.
12 Holland and Belgium.
13 Switzerland.
14 Spain and Portugal.
15 Italy.
16 Sweden, Norway, and Denmark.
17 German Empire.
18 Austria.
19 Russia in Europe.
20 Turkey in Europe, and Greece.
21 India.
22 Persia, Afghanistan, and Beloochistan.
23 Turkey in Asia.
24 Chinese Empire and Japan.
25 Arabia, Egypt, and Nubia.
26 Palestine.
27 Dominion of Canada.
28 United States.
29 West Indies.
30 Australia.
31 New South Wales, Victoria, and South Australia.
32 New Zealand.

THE CROWN ATLAS, consisting of 32 Maps, on Guards, with Index, 8vo, cloth lettered, ... 2 6

THE NATIONAL ATLAS, consisting of 32 Maps, 4to, with a Copious Index, cloth lettered, ... 2 6

London, Edinburgh, and Herriot Hill Works, Glasgow.

COLLINS' SERIES OF SCHOOL ATLASES—CONTINUED.

MODERN GEOGRAPHY—Imperial Series.

s. d.

THE SELECTED ATLAS, consisting of 16 Maps, Imperial 4to, 11 by 13 inches, Stiff Cover, ... 1 6

1 The Hemispheres.
2 Europe.
3 Asia.
4 Africa.
5 North America.
6 South America.
7 England and Wales.
8 Scotland.
9 Ireland.
10 Southern and Central Europe.
11 India.
12 Canada.
13 United States.
14 Australia.
15 New Zealand.
16 Palestine.

THE PORTABLE ATLAS, consisting of 16 Maps, folded Imperial 8vo, cloth lettered, ... 2 0

THE ADVANCED ATLAS, consisting of 32 Maps, Imperial 4to, cloth lettered, ... 3 6

1 Eastern and Western Hemispheres.
2 The World, (Mercator's Projection.)
3 Europe.
4 Asia.
5 Africa.
6 North America.
7 South America.
8 England and Wales.
9 Scotland.
10 Ireland.
11 France.
12 Holland and Belgium.
13 Switzerland.
14 Spain and Portugal.
15 Italy.
16 Sweden and Norway, Denmark and the Baltic.
17 German Empire.
18 Austria.
19 Russia.
20 Turkey in Europe, and Greece.
21 India.
22 Persia, Afghanistan, and Beloochistan.
23 Turkey in Asia.
24 Chinese Empire, and Japan.
25 Arabia, Egypt, Nubia, and Abyssinia.
26 Palestine.
27 Dominion of Canada.
28 United States.
29 West Indies and Central America.
30 Australia.
31 Victoria, New South Wales, and South Australia.
32 New Zealand.

THE ACADEMIC ATLAS, consisting of 32 Maps, Imperial 4to, with a Copious Index, cloth lettered, ... 5 0

THE STUDENT'S ATLAS, consisting of 32 Maps, and 6 Ancient Maps, with a Copious Index, Imperial 8vo, cloth lettered, ... 6 0

33 Ancient Greece.
34 Ancient Roman Empire.
35 Britain under the Romans.
36 Britain under the Saxons.
37 Historical Map of the British Islands from A.D. 1066.
38 France and Belgium, illustrating British History.

THE COLLEGIATE ATLAS, consisting of 32 Modern, 16 Historical, and 2 Ancient Maps, mounted on Guards, with a Copious Index, Imperial 8vo, cloth lettered, ... 7 6

THE INTERNATIONAL ATLAS, consisting of 32 Modern, 16 Historical, and 14 Maps of Classical Geography, with Descriptive Letterpress on Historical Geography by W. F. Collier, LL.D.; and on Classical Geography by L. Schmitz, LL.D., with Copious Indices, Imperial 8vo, cloth mounted on Guards, ... 10 6